Griffiths Nuclear & Particle 1st ed.

CHAPTER 6

Section 6.1

Lifetimes & Cross Sections

Presentation 11/2 Erin M. Cooper

Notes → Typewritten copy w/ handwritten notes & appended handwritten notes along w/ copy of pgs. 109-10 in Goldstein 3rd ed. Ch. 3 for derivation of impact parameter in Rutherford scattering.

Chapter 6

The Feynman Calculus

In this chapter we begin the quantitative formulation of elementary particle dynamics, which amounts, in practice, to the calculation of decay rates (Γ) and scattering cross sections (σ). The procedure involves two distinct parts: (1) evaluation of the relevant Feynman diagrams to determine the "amplitude" ($\mathcal{M}$) for the process in question, and (2) insertion of $\mathcal{M}$ into Fermi's "Golden Rule" to compute Γ or σ, as the case may be. To avoid distracting algebraic complications, I introduce here a simplified model. Realistic theories—QED, QCD, and GWS—are developed in succeeding chapters. If you like, Chapter 6 can be read immediately after Chapter 3. Study it with scrupulous care, or what follows will be unintelligible.

6.1 LIFETIMES AND CROSS SECTIONS

Present Mon 11/2

As I mentioned in the Introduction, we have three experimental probes of elementary particle interactions: bound states, decays and scattering. Nonrelativistic quantum mechanics (in Schrödinger's formulation) is particularly well adapted to handle bound states, which is why we used it, as far as possible, in Chapter 5. By contrast, the relativistic theory (in Feynman's formulation) is especially well suited to describe decays and scattering. In this chapter I'll introduce the basic ideas and strategies of the Feynman "calculus"; in subsequent chapters we will use it to develop the theories of strong, electromagnetic, and weak interactions.

To begin with, we must decide what physical quantities we would like to calculate. In the case of decays, the item of greatest interest is the *lifetime* of the particle in question. What precisely do we mean by the lifetime of, say, the muon? We have in mind, of course, a muon at *rest*; a moving muon lasts longer (from our perspective) because of time dilation. But even stationary muons don't all last the same amount of time, for there is an intrinsically random element in the decay process. We cannot hope to calculate the lifetime of any particular muon; rather, what we are after is the *average* (or "mean") lifetime, τ, of the muons in any large sample. Now, elementary particles have no memories, so the probability of a given muon decaying in the next microsecond is independent of how long ago that muon was created. (It's quite different in biological systems: An 80-year-old man is much more likely to die in the next year than is a 20-year-old, and his body shows the signs of eight decades of wear and tear. But all muons are identical, regardless of when they were produced; from an actuarial point of view they're all on an equal footing.) The critical parameter, then, is the *decay rate*, Γ, the *probability per unit time* that any given muon will disintegrate. If we had a large collection of muons, say, $N(t)$, at time t, then $N\,\Gamma dt$ of them would decay in the next instant dt. This would, of course, *decrease* the number remaining:

$$dN = -\Gamma N dt \tag{6.1}$$

It follows that

$$N(t) = N(0)e^{-\Gamma t} \tag{6.2}$$

see attached for notes & derivation

Evidently, the number of particles left decreases exponentially with time. As you can check for yourself (Problem 6.1), the mean lifetime is simply the reciprocal of the decay rate:

$$\tau = \frac{1}{\Gamma} \tag{6.3}$$

Actually, most particles can decay by several routes. The π^+, for instance, usually decays into $\mu^+ + \nu_\mu$, but sometimes goes into $e^+ + \nu_e$; occasionally a π^+ decays into $\mu^+ + \nu_\mu + \gamma$, and they have even been known to go into $e^+ + \nu_e + \pi^0$. In such circumstances the *total* decay rate is the sum of the individual decay rates:

$$\Gamma_{tot} = \sum_{i=1}^{n} \Gamma_i \tag{6.4}$$

and the lifetime of the particle is the reciprocal of Γ_{tot}:

$$\tau = 1/\Gamma_{tot} \tag{6.5}$$

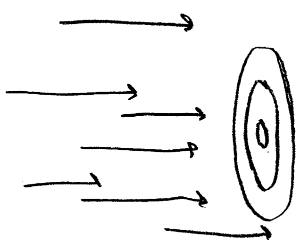
In addition to τ, we want to calculate the various *branching ratios*, that is, the fraction of all particles of the given type that decay by each mode. Branching ratios are determined by the decay rates:

$$\text{Branching ratio for the } i\text{th decay mode} = \Gamma_i/\Gamma_{tot} \tag{6.6}$$

For decays, then, $\boxed{\text{the essential problem is to calculate the decay rate } \Gamma_i}$ for each mode; from there it is an easy matter to obtain the lifetime and branching ratios.

How about scattering? What quantity should the experimentalist measure and the theorist calculate? If we were talking about an *archer* aiming at a "bull's-eye," the parameter of interest would be the *size of the target*, or more precisely the cross-sectional area it presents to a stream of incoming arrows. In a crude sense, the same goes for elementary particle scattering: If you fire a stream of electrons into a tank of hydrogen (which is essentially a collection of protons) the parameter of interest is the size of the proton—the cross-sectional area σ it presents to the incident beam. The situation is more complicated than in archery, however, for several reasons. First of all the target is "soft"; it's not a simple case of "hit-or-miss," but rather "the closer you come the greater the deflection." Nevertheless, it is still possible to define an "effective" cross section; I'll show you how in the next paragraph. Second, the cross section depends on the nature of the "arrow" as well as the structure of the "target." Electrons scatter off hydrogen more sharply than neutrinos and less so than pions, because different interactions are involved. It depends too, on the *outgoing* particles; if the energy is high enough we can have not only *elastic* scattering ($e + p \rightarrow e + p$), but a variety of *inelastic* processes, such as $e + p \rightarrow e + p + \gamma$, or $e + p + \pi^0$, or even in principle, $\nu_e + \Lambda$. Each one of these has its own ("exclusive") scattering cross section, σ_i (for process i). In some experiments, however, the final products are not examined, and we are interested only in the *total* ("inclusive") cross section.

relavent parameter is cross section target presents to incoming stream of particles.



$$\sigma_{tot} = \sum_{i=1}^{n} \sigma_i \tag{6.7}$$

Finally, each cross section typically depends on the *velocity* of the incident particle. At the most naïve level we might expect the cross section to be proportional to the amount of time the incident particle spends in the vicinity of the target, which is to say that σ should be inversely proportional to υ. But this behavior is dramatically altered in the neighborhood of a "resonance"—a special energy at which the particles involved "like" to interact, forming a short-lived semibound state before breaking apart. Such "bumps" in the graph of σ versus υ (or, as it is more commonly plotted, σ versus E) are in fact the principal means by which short-lived particles are discovered (see Fig. 4.6). So, unlike the archer's target, there's a lot of physics in an elementary particle cross section.

Let's go back, now, to the question of what we mean by a "cross section" when the target is "soft." Suppose a particle (maybe an electron) comes along, encounters some kind of potential (perhaps the Coulomb potential of a stationary proton), and scatters off at an angle θ. This *scattering angle* is a function of the *impact parameter b*, the distance by which the incident particle would have missed the scattering center, had it continued on its original trajectory (Fig. 6.1). Ordinarily, the smaller the impact parameter, the larger the deflection, but the actual functional form of θ(b) depends on the particular potential involved.

EXAMPLE 6.1 Hard-Sphere Scattering $\longrightarrow$ After $d\sigma/d\Omega$

Suppose the particle bounces elastically off a sphere of radius R. From Figure 6.2, we have

$$b = R\sin\alpha, \qquad 2\alpha + \theta = \pi \quad \rightarrow \quad \alpha = \frac{\pi}{2} - \frac{\theta}{2}$$

Thus $$\sin\alpha = \sin(\pi/2 - \theta/2) = \cos(\theta/2)$$

and hence $$b = R\cos(\theta/2) \quad \text{or} \quad \theta = 2\cos^{-1}(b/R)$$

This is the relation between θ and b for classical hard-sphere scattering.

If the particle comes in with an impact parameter between b and $b + db$, it will emerge with a scattering angle between $\theta + d\theta$. More generally, if it passes through an infinitesimal *area* $d\sigma$, it will scatter into a corresponding *solid angle* $d\Omega$ (Fig. 6.3). Naturally, the larger we make $d\sigma$, the larger $d\Omega$ will be. The proportionality factor is called the *differential scattering cross section, D*:

through $d\sigma$ $\longrightarrow$ Scatter into $d\Omega$

proportionality $\frac{d\sigma}{d\Omega}$ = differential scattering cross section

$$d\sigma = D(\theta)d\Omega \tag{6.8}$$

In principle, D might depend on the azimuthal angle ϕ; however, most potentials of interest are spherically symmetrical, in which case the differential cross section depends only on θ (or, if you prefer, on b). By the way, the notation, D, is my own; most people call it simply $d\sigma/d\Omega$, and in the rest of the book I'll revert to the standard terminology. The name "differential cross section" is poorly chosen; it's not a differential at all, in the mathematical sense (the words would apply more naturally to $d\sigma$ than to $d\sigma/d\Omega$).

Now, from Figure 6.3 we see that

differential cross section

$$d\sigma = |b\,db\,d\phi|, \qquad d\Omega = |\sin\theta\,d\theta d\phi| \qquad \frac{d\sigma}{d\Omega} = D \tag{6.9}$$

(Areas and solid angles are intrinsically positive, hence the absolute value signs.) Accordingly,

$$D(\theta) = \frac{d\sigma}{d\Omega} = \left|\frac{b}{\sin\theta}\left(\frac{db}{d\theta}\right)\right| \tag{6.10}$$

EXAMPLE 6.2

In the case of hard-sphere scattering, Example 6.1, we find

$$\frac{db}{d\theta} = -\frac{R}{2}\sin\left(\frac{\theta}{2}\right)$$

$b = R\cos\frac{\theta}{2}$

$db = -\frac{R}{2}\sin\frac{\theta}{2}\,d\theta$

and hence

$$D(\theta) = \frac{Rb\sin(\theta/2)}{2\sin\theta} = \frac{R^2}{2}\frac{\cos(\theta/2)\sin(\theta/2)}{\sin\theta} = \frac{R^2}{4}$$

$\sin 2\theta = 2\sin\theta\cos\theta$

Finally, the *total* cross section is the integral of $d\sigma$ over all solid angles: $\Rightarrow \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{1}{2}\sin\theta$

$$\sigma = \int d\sigma = \int D(\theta)d\Omega \tag{6.11}$$

EXAMPLE 6.3

For hard-sphere scattering

$\sigma = \int d\sigma = \int D(\theta)\,d\Omega = \int \frac{R^2}{4}d\Omega = \int_0^{2\pi}\int_0^{\pi} \frac{R^2}{4}\sin\theta\,d\theta\,d\phi$

$$\sigma = \int \frac{R^2}{4}d\Omega = \pi R^2$$

$= \frac{2\pi}{4}R^2\int \sin\theta\,d\theta = -\frac{\pi R^2}{2}\left(\cos\theta\right)\Big|_0^{\pi} = -\frac{\pi R^2}{2}(-1-1)$

$= \pi R^2$

which is, of course, the total cross section the sphere presents to an incoming beam: Any particles *within* this area will scatter, any *outside* will pass unaffected.

As Example 6.3 indicates, the formalism developed here is consistent with our naïve sense of the term "cross section," in the case of a "hard" target; its virtue is that it applies as well to "soft" targets, which do not have sharp edges.

EXAMPLE 6.4 Rutherford Scattering

A particle of charge q_1 scatters off a stationary particle of charge q_2. In classical mechanics the formula relating the impact parameter to the scattering angle is

$$b = \frac{q_1 q_2}{2E}\cot(\theta/2)$$

see included copies of Goldstien, 3rd ed. pgs 109-110 for derivation of impact parameter

where E is the initial kinetic energy of the incident charge. The differential cross section is therefore

$$D(\theta) = \left(\frac{q_1 q_2}{4E\sin^2(\theta/2)}\right)^2$$

In this case the total cross section is actually *infinite*:

$$\sigma = 2\pi\left(\frac{q_1 q_2}{4E}\right)^2 \int_0^{\pi} \frac{1}{\sin^4(\theta/2)}\sin\theta d\theta = \infty$$

Suppose now, that we have a *beam* of incoming particles, with uniform *luminosity* $\mathcal{L}$ ($\mathcal{L}$ is the number of particles passing down the line per unit time, per unit area). Then $dN = \mathcal{L}\, d\sigma$ is the number of particles per unit time passing through area $d\sigma$, and hence also the number per unit time scattered into solid angle $d\Omega$:

$$dN = \mathcal{L}d\sigma = \mathcal{L}D(\theta)d\Omega$$

It follows that

see Attached for derivation

$$\frac{d\sigma}{d\Omega} = D(\theta) = \frac{1}{\mathcal{L}}\frac{dN}{d\Omega} \tag{6.12}$$

This is frequently a more convenient way to think of the differential cross section: It is the number of particles per unit time scattered into solid angle $d\Omega$, divided by $d\Omega$ and by the luminosity. (Or, as accelerator physicists like to put it, "the event rate is the cross section times the luminosity.")

DECAY RATES & LIFETIMES

Γ - probability per unit time that any given particle would decay
$N(t)$ - total # particles at time t, not yet decayed
$N\Gamma dt$ - number of particles that would decay in next instant
→ since this would <u>decrease</u> $N(t)$

$$dN = -\Gamma N dt \qquad (6.1)$$

By integration

$$\int \frac{dN}{N} = \int -\Gamma dt$$

so $\ln N = -\Gamma t + C$

where $C = \ln(N(0))$

EXPONENTIATING BOTH SIDES:

$$e^{\ln N} = e^{-\Gamma t + C}$$

$$N = e^{\ln(N(0))} e^{-\Gamma t}$$

$$N = N(0)\, e^{-\Gamma t} \qquad (6.2)$$

lifetime is ~~time it takes for sample to be reduced by a factor of e~~
~~lifetime, $\tau = 1/\Gamma$, so~~
~~$N(\tau) = N(0)\, e^{-\Gamma/\Gamma} = N(0)\, e^{-1} = \frac{1}{e} N(0)$~~

~~see next page~~ see attached for derivation of τ

often particles decay by many routes, eg

$\pi^+ \longrightarrow \mu^+ + \nu_\mu$ usually
$\longrightarrow e^+ + \nu_e$ sometimes
$\longrightarrow \mu^+ + \nu_\mu + \gamma$ occasionally
$\longrightarrow e^+ + \nu_e + \pi^0$ have been known to

in this case decay rate is sum of individuals

$$\Gamma_{tot} = \sum_{i=1}^{n} \Gamma_i \qquad \left(N = N(0) \prod_{i=1} e^{-\Gamma_i t} \right)$$

BRANCHING RATIOS -

BRANCHING RATIO

- fraction of particles that decay by each mode -

determined by ratio of decay rates

$BR_{i\text{th mode}} = \Gamma_i / \Gamma_{tot}$

Derivation of the mean lifetime

Given an assembly of elements, the number of which decreases ultimately to zero, the **mean lifetime**, τ, (also called simply the **lifetime**) is the expected value of the amount of time before an object is removed from the assembly. Specifically, if the *individual lifetime* of an element of the assembly is the time elapsed between some reference time and the removal of that element from the assembly, the mean lifetime is the arithmetic mean of the individual lifetimes.

Starting from the population formula

$$N = N_0 e^{-\lambda t},$$

we firstly let c be the normalizing factor to convert to a probability space:

$$1 = \int_0^\infty c \cdot N_0 e^{-\lambda t}\, dt = c \cdot \frac{N_0}{\lambda}$$ $cN_0\left(\frac{1}{\lambda}\right.$

or, on rearranging,

$$c = \frac{\lambda}{N_0}.$$

We see that exponential decay is a scalar multiple of the exponential distribution (i.e. the individual lifetime of a each object is exponentially distributed), which has a well-known expected value. We can compute it here using integration by parts.

$$\tau = \langle t \rangle = \int_0^\infty t \cdot c \cdot N_0 e^{-\lambda t}\, dt = \int_0^\infty \lambda t e^{-\lambda t}\, dt = \frac{1}{\lambda}.$$

$N = N_0 e^{-\Gamma t}$

in probability space

$$1 = \int_0^\infty c N_0 e^{-\Gamma t} dt = -\frac{cN_0}{\Gamma}\left(e^{-\Gamma t}\right)_0^\infty = -\frac{cN_0}{\Gamma}(0-1) = \frac{cN_0}{\Gamma}$$

So then $C = \Gamma / N_0$

and

$$\tau = \langle t \rangle = \int_0^\infty t \frac{\Gamma}{\cancel{N_0}} \cancel{N_0} e^{-\Gamma t} dt = \Gamma\left(\cancel{-\frac{t}{\Gamma} e^{-\Gamma t}\Big|_0^\infty} - \int_0^\infty \left(\frac{-1}{\Gamma}\right) e^{-\Gamma t} dt\right)$$

$$= \cancel{\Gamma}\left(\cancel{\frac{1}{\Gamma}}\left(\frac{-1}{\Gamma}\left(e^{-\Gamma t}\right)_0^\infty\right)\right) = \frac{1}{\Gamma}$$

let $u = t$ $dv = e^{-\Gamma t} dt$

$du = dt$ $v = \frac{-1}{\Gamma} e^{-\Gamma t}$

$$\lim_{t\to\infty} \frac{t}{e^{\Gamma t}} = \lim_{t\to\infty} \frac{1}{\Gamma e^{\Gamma t}} = 0$$

force and the incident velocity. If v_0 is the incident speed of the partic

$$l = mv_0 s = s\sqrt{2mE}.$$

ce E and s are fixed, the angle of scattering Θ is then determined u
r the moment, it will be assumed that different values of s cannot le
ne scattering angle. Therefore, the number of particles scattered in
gle $d\Omega$ lying between Θ and $\Theta + d\Theta$ must be equal to the numb
ident particles with impact parameter lying between the correspond
$\cdot\, ds$:

$$2\pi I s|ds| = 2\pi\sigma(\Theta) I \sin\Theta\, |d\Theta|.$$

solute value signs are introduced in Eq. (3.91) because numbers of
st of course always be positive, while s and Θ often vary in opposite
is considered as a function of the energy and the corresponding
le,

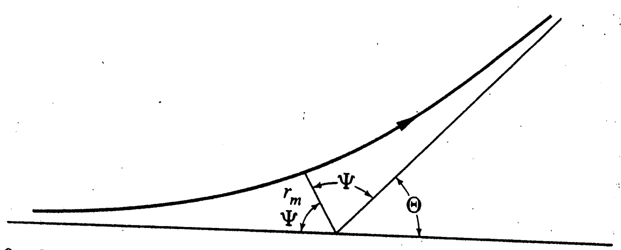
$$s = s(\Theta, E),$$

1 the dependence of the differential cross section on Θ is given by

$$\sigma(\Theta) = \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right|$$

A formal expression for the scattering angle Θ as a function of s
ly obtained from the orbit equation, Eq. (3.36). Again, for simplicity
sider the case of purely repulsive scattering (cf. Fig. 3.20). As the o
ymmetric about the direction of the periapsis, the scattering angle is

$$\Theta = \pi - 2\Psi,$$



JRE 3.20 Relation of orbit parameters and scattering angle in an exampl
cattering.

at this point in the formulation that classical and quantum mechanics part comp
undamentally characteristic of quantum mechanics that we cannot unequivocally
ory of any particular particle. We can only give probabilities for scattering in variou

Determination of impact parameter in Rutherford scattering (charged particles in a Coulomb field.

where Ψ is the angle between the direction of the incoming asymptote and the periapsis (closest approach) direction. In turn, Ψ can be obtained from Eq. (3.36) by setting $r_0 = \infty$ when $\theta_0 = \pi$ (the incoming direction), whence $\theta = \pi - \Psi$ when $r = r_m$, the distance of closest approach. A trivial rearrangement then leads to

$$\Psi = \int_{r_m}^{\infty} \frac{dr}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{1}{r^2}}}. \tag{3.95}$$

Expressing l in terms of the impact parameter s (Eq. (3.90)), the resultant expression for $\Theta(s)$ is

$$\Theta(s) = \pi - 2\int_{r_m}^{\infty} \frac{s\,dr}{r\sqrt{r^2\left(1 - \frac{V(r)}{E}\right) - s^2}}, \tag{3.96}$$

or, changing r to $1/u$

$$\Theta(s) = \pi - 2\int_{0}^{u_m} \frac{s\,du}{\sqrt{1 - \frac{V(u)}{E} - s^2u^2}}. \tag{3.97}$$

Equations (3.96) and (3.97) are rarely used except for direct numerical computation of the scattering angle. However, when an analytic expression is available for the orbits, the relation between Θ and s can often be obtained almost by inspection. An historically important illustration of such a procedure is the repulsive scattering of charged particles by a Coulomb field. The scattering force field is that produced by a fixed charge $-Ze$ acting on the incident particles having a charge $-Z'e$ so that the force can be written as

$$f = \frac{ZZ'e^2}{r^2},$$

i.e., a repulsive inverse-square law. The results of Section 3.7 can be taken over here with no more change that writing the force constant as

$$k = -ZZ'e^2. \tag{3.98}$$

The energy E is greater than zero, and the orbit is a hyperbola with the eccentricity given by*

$$\epsilon = \sqrt{1 + \frac{2El^2}{m(ZZ'e^2)^2}} = \sqrt{1 + \left(\frac{2Es}{ZZ'e}\right)^2} \tag{3.99}$$

where use has been made of Eq. (3.90). If θ' in Eq. (3.55) is chosen to be π, periapsis corresponds to $\theta = 0$ and the orbit equation becomes

*To avoid confusion with the electron charge e, the eccentricity will temporarily be denoted by ϵ.

Goldstein, 3rd ed.

$$\frac{1}{r} = \frac{mZZ'e}{l^2}(\epsilon \cos\theta - 1). \qquad (3.100$$

This hyperbolic orbit equation has the same form as the elliptic orbit equa tion (3.56) except for a change in sign. The direction of the incoming asymptote Ψ, is then determined by the condition $r \to \infty$:

$$\cos\Psi = \frac{1}{\epsilon}$$

or, by Eq. (3.94),

$$\sin\frac{\Theta}{2} = \frac{1}{\epsilon}.$$

Hence,

$$\cot^2\frac{\Theta}{2} = \epsilon^2 - 1,$$

and using Eq. (3.99)

$$\cot\frac{\Theta}{2} = \frac{2Es}{ZZ'e}$$

The desired functional relationship between the impact parameter and the scatte ing angle is therefore

$$s = \frac{ZZ'e^2}{2E}\cot\frac{\Theta}{2}, \qquad (3.10$$

so that on carrying through the manipulation required by Eq. (3.93), we find th $\sigma(\Theta)$ is given by

$$\sigma(\Theta) = \frac{1}{4}\left(\frac{ZZ'e^2}{2E}\right)^2 \csc^4\frac{\Theta}{2}. \qquad (3.10$$

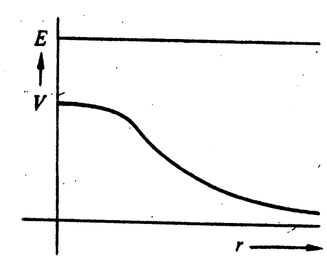
Equation (3.102) gives the famous Rutherford scattering cross section, or inally derived by Rutherford for the scattering of α particles by atomic nucl Quantum mechanics in the nonrelativistic limit yields a cross section identic with this classical result.

In atomic physics, the concept of a *total scattering cross section* σ_T, defin as

$$\sigma_T = \int_{4\pi} \sigma(\mathbf{\Omega})\, d\Omega = 2\pi \int_0^{\pi} \sigma(\Theta)\sin\Theta\, d\Theta,$$

obtain an infinite result! The physi cult to discern. From its definition t cles scattered in all directions per u Coulomb field is an example of a "lo The very small deflections occur onl eters. Hence, all particles in an inc scattered to some extent and must be It is therefore clear that the infinite field; it occurs in classical mechani from zero at all distances, no matter i.e., is zero beyond a certain distanc Physically, such a cut-off occurs for the presence of the atomic electrons cancel its charge outside the atom.

In Rutherford scattering, the scat tion of the impact parameter s. From infinity, Θ increases monotonically zero. However, other types of behavi some modification in the prescriptio For example, with a repulsive poter nature shown in Fig. 3.21(a), it is ea sus s may behave as indicated in F the impact parameter, as noted above distances from the center of force an extreme, for $s = 0$, the particle trav and if the energy is greater than the on through the center without being



(a)

FIGURE 3.21 Repulsive nonsingular scattering angle Θ versus impact paramet

Rutherford Scattering

$$b = \frac{q_1 q_2}{2E} \cot(\theta/2)$$

$$D(\theta) = \left| \frac{b}{\sin\theta} \frac{db}{d\theta} \right|$$

$$\frac{db}{d\theta} = \frac{q_1 q_2}{2E} \frac{d}{d\theta}\left(\cot(\theta/2)\right)$$

$$\frac{d}{d\theta} \cot(\theta/2) = \frac{d}{d\theta}\left(\frac{\cos(\theta/2)}{\sin(\theta/2)}\right) = -\frac{1}{2}\frac{\sin(\theta/2)}{\sin(\theta/2)} - \frac{\cos(\theta/2) \times \cos(\theta/2)}{\sin^2(\theta/2)}$$

$$= -\frac{1}{2}\left(1 + \frac{\cos^2\theta/2}{\sin^2\theta/2}\right) = -\frac{1}{2}\left(\frac{\sin^2\theta/2 + \cos^2\theta/2}{\sin^2\theta/2}\right)$$

$$= -\frac{1}{2\sin^2(\theta/2)}$$

so then

$$D(\theta) = \left|\frac{b}{\sin\theta} \frac{db}{d\theta}\right| = \left(\frac{q_1 q_2}{2E}\right)^2 \frac{\cot\theta/2}{\sin\theta} \frac{1}{2\sin^2(\theta/2)}$$

see next page

Trig Identities

$$\cot\theta/2 = \frac{\sin\theta}{1-\cos\theta}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\Rightarrow 2\sin^2\theta/2 = 1-\cos\theta$$

$$\cot\theta/2 = \frac{\sin\theta}{1-\cos\theta} = \frac{\sin\theta}{2\sin^2\theta/2}$$

$$D(\theta) = \left|\frac{b}{\sin\theta}\frac{db}{d\theta}\right|$$

$$= \left(\frac{q_1 q_2}{2E}\right)^2 \frac{\cot\theta/2}{\sin\theta}\frac{1}{2\sin^2\theta/2}$$

$$= \left(\frac{q_1 q_2}{2E}\right)^2 \frac{\cancel{\sin\theta}}{2\sin^2\theta/2}\frac{1}{\cancel{\sin\theta}\,2\sin^2\theta/2}$$

$$= \left(\frac{q_1 q_2}{4E}\frac{1}{\sin^2\theta/2}\right)^2$$

$$\sigma = \int D(\theta)\,d\Omega = \int_0^{2\pi}\int_0^{\pi} D(\theta)\sin\theta\,d\theta\,d\phi$$

$$= 2\pi\left(\frac{q_1 q_2}{4E}\right)^2 \int_0^{\pi}\frac{\sin\theta\,d\theta}{\sin^4(\theta/2)}$$

$$\int_0^{\pi}\frac{\sin\theta\,d\theta}{\sin^4(\theta/2)} = \int_0^{\pi}\frac{\sin\theta\,d\theta}{\frac{(1-\cos\theta)^2}{4}} = 4\int_0^{\pi}\frac{\sin\theta}{(1-\cos\theta)^2}d\theta$$

let $u = -\cos\theta$ $\quad du = \sin\theta\,d\theta$

$u(0) = -1 \quad u(\pi) = 1$

then $\rightarrow$ $$4\int_{-1}^{1}\frac{du}{(1+u)^2} = 4\left(\frac{-1}{1+u}\right)\Bigg|_{-1}^{1} = 4\left[\frac{-1}{2} + \frac{1}{0}\right] \rightarrow \infty$$

* EM FORCE HAS infinite RANGE

Beam of particles w/ uniform luminosity, $\mathcal{L}$

$\mathcal{L}$ - luminosity

- # particles per unit time per unit area

so $dN = \mathcal{L}\, d\sigma$ is # particles per unit time passing through $d\sigma$

so now

$$dN = \mathcal{L}\, d\sigma = \mathcal{L}\, D(\theta)\, d\Omega$$

and the differential scattering cross section is

$$\frac{d\sigma}{d\Omega} = D(\theta) = \frac{1}{\mathcal{L}} \frac{dN}{d\Omega}$$

so then differential cross section is # particles per unit time scattered into $d\Omega$ divided by $\mathcal{L}\, d\Omega$

→ Event rate is cross section × luminosity

$$\frac{dN}{d\Omega} = \mathcal{L} \frac{d\sigma}{d\Omega}$$

$$\cancel{dN = \mathcal{L}\, d\sigma = -\Gamma N\, dt}$$

$$\cancel{\Gamma = -\frac{\mathcal{L}}{N} \frac{d\sigma}{dt}}$$