Energy - Momentum Relation

$p_\mu p^\mu - m^2c^2 = 0 \qquad p_\mu \to i\partial_\mu$

$\Rightarrow (\partial_\mu \partial^\mu + m^2)\phi = 0 \ , \ \underbrace{\phi = A e^{ip.x}}_{\text{Scalar wave function}}$ Klein Gordon

We are looking for a relativistic eq. with solutions that have 2 component structure for both particle & antiparticle

Dirac's goal was to write an eq. which, unlike K-G eq., was linear in time.

In order to be covariant it must be also linear in ∇. The general form is:

$(\underline{\alpha}.\underline{p} + \beta m)\psi = H\psi \qquad \alpha_i \quad i = 1,2,3$

β & α_i $(i=1,2,3)$ are determined by using the fact that it should satisfy:

$(\underline{p}^2 + m^2)\psi = H^2\psi \quad \longrightarrow \quad H^2 = \underline{p}^2 + m^2$

$H^2\psi = (\underline{\alpha}.\underline{p} + \beta m)^2\psi$

$= (\alpha_i p_i + \beta m)(\alpha_j p_j + \beta m)\psi$

$= (\alpha_i \alpha_j p_i p_j + m(\alpha_i\beta + \beta\alpha_i)p_i + \beta^2 m^2)\psi$

$= [(\underbrace{\alpha_i\alpha_i}_{1} \underbrace{p_i p_i}_{p^2})_{i=j} + \underbrace{\tfrac{1}{2}(\alpha_i\alpha_j + \alpha_j\alpha_i) p_i p_j}_{0 \ (i \neq j)} + \underbrace{m(\alpha_i\beta + \beta\alpha_i)p_i}_{0} + \underbrace{\beta^2}_{1} m^2]$

or $i > j$

Compare to the energy-momentum relation:

$\alpha_i^2 = 1$

$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \quad i \neq j \Rightarrow$

$\alpha_i \beta + \beta \alpha_i = 0$

$\beta^2 = 1$

$\{\alpha_i, \alpha_i\} = 2$

$\{\alpha_i, \alpha_j\} = 0$

$\{\alpha_i, \beta\} = 0$

$\{\beta, \beta\} = 2$

The smallest matrices that satisfy this are 4x4 dim.

$$\underline{\alpha} = \begin{pmatrix} 0 & \underline{\sigma} \\ \underline{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} \underline{I} & 0 \\ 0 & -\underline{I} \end{pmatrix} \qquad d_{\substack{\text{Dirac} \\ \text{Matrices}}} = 2^{[D/2]}$$

Dirac-Pauli rep.

$\{\sigma_i, \sigma_j\} = 2\delta_{ij}$

$\sigma_i^\dagger = \sigma_i$

$\det \sigma_i = -1$

$\mathrm{Tr}(\sigma_i) = 0$

Covariant form of the Dirac Eqn.

$\times\beta \quad (\underline{\alpha}\cdot\underline{p} + \beta m)\psi = i\dfrac{\partial\psi}{\partial t}$

$(\beta \alpha_i p_i + \beta^2 m)\psi = i\beta\dfrac{\partial\psi}{\partial t}$

$(-i\beta\underline{\alpha}\cdot\nabla + 1m)\psi = i\beta\dfrac{\partial\psi}{\partial t}$

now define: $\gamma^0 = \beta$

$\gamma^i = \beta\alpha_i \quad (i=1,2,3)$

$\gamma^\mu = (\beta, \beta\underline{\alpha})$

$$(-i\underbrace{\beta\alpha_i}_{\gamma^i}\partial_i + 1m)\psi = i\underbrace{\beta}_{\gamma^0}\frac{\partial\psi}{\partial t}$$

$$(i\gamma^0\partial_0 + i\gamma^i\partial_i - m)\psi = 0$$

$$\Rightarrow (i\gamma^\mu\partial_\mu - m)\psi = 0 \qquad \text{Dirac Eq. covariant form.}$$

4 differential eqn.s

$$
\left.\begin{array}{l|l}
\beta\cdot & \alpha^i\alpha^i + \alpha^i\alpha^i = 2 \longrightarrow \overbrace{\beta\alpha^i}^{\gamma^i}\,\overbrace{(-\beta\alpha^i)}^{-\gamma^i} + \beta\alpha^i(-\beta\alpha^i) = 2 \\
 & \qquad\qquad \gamma^i\gamma^i + \gamma^i\gamma^i = -2 \\
\beta^2\alpha & \alpha^i\alpha^j + \alpha^j\alpha^i = 0 \quad i\neq j \longrightarrow \beta\alpha^i(\beta\alpha^j) + \beta\alpha^j(\beta\alpha^i) = 0 \quad i\neq j \\
 & \qquad\qquad \gamma^i\gamma^j + \gamma^j\gamma^i = 0 \quad j\neq i \\
\beta\alpha & \alpha^i\beta + \beta\alpha^i = 0 \longrightarrow (\beta\alpha^i)\underbrace{\beta}_{\gamma^0} + \beta(\beta\alpha^i) = 0 \\
 & \qquad\qquad \gamma^i\gamma^0 + \gamma^0\gamma^i = 0 \\
 & \beta\beta + \beta\beta = 2 \longrightarrow \beta\beta + \beta\beta = 2 \\
 & \qquad\qquad \gamma^0\gamma^0 + \gamma^0\gamma^0 = 2
\end{array}\right.
$$

putting it together: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$$\gamma^0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \Rightarrow \boxed{(\gamma^0)^\dagger = \gamma^0} * , \quad \boxed{(\gamma^0)^2 = I}$$

$$\gamma^k = \beta\alpha^k \Rightarrow (\gamma^k)^\dagger = (\alpha^k)^\dagger\beta^\dagger = \alpha^k\beta = -\beta\alpha^k = \boxed{-\gamma^k} **$$

(σ_i are Hermitian)

$$\boxed{(\gamma^k)^2 = (\beta\alpha^k\beta\alpha^k) = -\beta(\alpha^k)^2\beta = -\beta^2 = -I}$$

$(\alpha^i)^2 = 1$

$$* \ \& \ ** \longrightarrow \boxed{(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0}$$

- Conserved Current of the Adjoint Eqn.

Hermitian Conjugate of the Dirac Eqn.:

$$\left[i\gamma^\mu(\partial_\mu\psi) - m\psi\right]^\dagger = 0$$

$$\Rightarrow -i(\partial_\mu\psi)^\dagger(\gamma^\mu)^\dagger - m\psi^\dagger = 0$$

$$\Rightarrow i(\partial_\mu\psi^\dagger)\gamma^0\gamma^\mu\gamma^0 + m\psi^\dagger = 0 \quad \leftarrow \times\gamma^0$$

$$\Rightarrow i(\partial_\mu\left[\psi^\dagger\gamma^0\right])\gamma^\mu + m(\left[\psi^\dagger\gamma^0\right]) = 0$$

Define : $\boxed{\bar{\psi} \equiv \psi^\dagger\gamma^0}$ adjoint spinor.

$\boxed{i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0}$ H.D.E. D.E. $\boxed{i\gamma^\mu\partial_\mu - m\psi = 0}$

to derive the continuity eqn,

$$\bar{\psi} \times (\text{D.E.}) = 0 \Rightarrow \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi = 0$$

$$i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi = 0$$

$$(\text{H.D.E.})\times\psi = 0 \Rightarrow i\partial_\mu\bar{\psi}\gamma^\mu\psi + m\bar{\psi}\psi = 0$$

- adding $\longrightarrow$ $\bar{\psi}\gamma^\mu\partial_\mu\psi + \partial_\mu\bar{\psi}\gamma^\mu\psi = 0$

$$\Rightarrow \partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0$$

$$\boxed{j^\mu \equiv -e\,\bar{\psi}\gamma^\mu\psi} \Rightarrow \partial_\mu j^\mu = 0$$

a conserved current.

Pauli-Weiskopf perscription.