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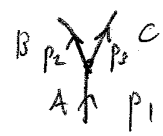
Sec. 6.6

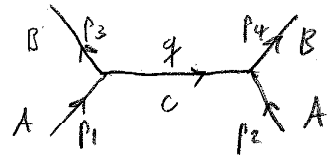
Lowest order amplitudes

For A decay, $A \to B + C$:



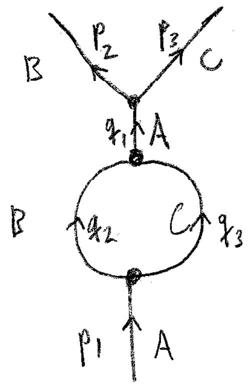
$$\mathcal{M} = g$$

For scattering, $\underset{p_1}{A} + \underset{p_2}{A} \to \underset{p_3}{B} + \underset{p_4}{B}$:



$$\mathcal{M} = \frac{g^2}{(p_4 - p_2)^2 - m_c^2 c^2}$$

One third-order correction for A decay



Follow Feynman Rules: to calculate amplitude $\mathcal{M}$:

<u>3 vertices</u>, <u>3 internal lines</u>

$$-i\mathcal{M} = (-ig)^3 \int \frac{i^3 (2\pi)^{4\cdot 3} \delta(p_1 - q_2 - q_3)\,\delta(q_2 + q_3 - q_1)\,\delta(q_1 - p_2 - p_3)}{(q_2^2 - m_B^2 c^2)(q_3^2 - m_C^2 c^2)(q_1^2 - m_A^2 c^2)\ (2\pi)^{4\cdot 3}}\, d^4q_1\, d^4q_2\, d^4q_3$$

$$= g^3 \int \frac{\delta(p_1 - q_2 - q_3)\,\delta(q_2 + q_3 - q_1)\,\delta(q_1 - p_2 - p_3)}{(q_2^2 - m_B^2 c^2)(q_3^2 - m_C^2 c^2)(q_1^2 - m_A^2 c^2)}\, d^4q_1\, d^4q_2\, d^4q_3$$

$$q_1 = p_2 + p_3$$

$$-i\mathcal{M} = \frac{g^3}{[(p_2+p_3)^2 - m_A^2 c^2]} \int \frac{\delta(p_1 - q_2 - q_3)\,\delta(q_2 + q_3 - p_2 - p_3)\, d^4q_2\, d^4q_3}{(q_2^2 - m_B^2 c^2)(q_3^2 - m_C^2 c^2)}$$

$$q_2 = p_1 - q_3$$

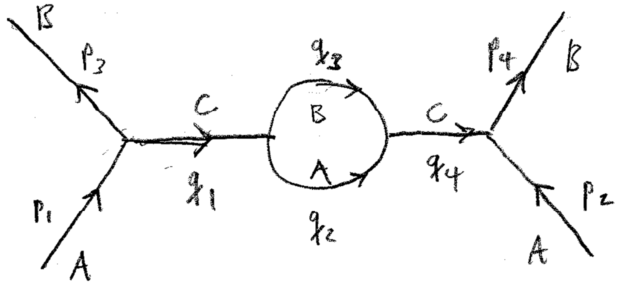
$$-i\mathcal{M} = \frac{g^3}{[(p_2+p_3)^2 - m_A^2 c^2]} \int \frac{\delta(p_1 - q_3 + q_3 - p_2 - p_3)\, d^4q_3}{[(p_1 - q_3)^2 - m_B^2 c^2](q_3^2 - m_C^2 c^2)}$$

Remove $(2\pi)^4 \delta(p_1 - p_2 - p_3)$ term;

$$-i\mathcal{M} = \frac{g^3/(2\pi)^4}{[(p_2+p_3)^2 - m_A^2 c^2]} \int \frac{d^4q}{[(p_1 - q)^2 - m_B^2 c^2](q^2 - m_C^2 c^2)}$$

The integral $\dfrac{g^3/(2\pi)^4}{[(p_2+p_3)^2 - m_A^2c^2]} \displaystyle\int \dfrac{d^4q}{[(p_1-q)^2 - m_B^2c^2](q^2 - m_C^2c^2)}$

is divergent. Similarly, for the scattering diagram



we obtain

$$\mathcal{M} = i\left(\frac{g}{2\pi}\right)^4 \frac{1}{[(p_1-p_3)^2 - m_C^2c^2]^2} \int \frac{d^4q}{[(p_1-p_3-q)^2 - m_A^2c^2](q^2 - m_B^2c^2)},$$

which also diverges when integrated over internal momentum q.

This is resolved by introducing a term

$\dfrac{-M^2c^2}{q^2 - M^2c^2}$ under the integral and through the application of several mathematical tricks, obtaining a finite term with no M dependence and a term logarithmic in M which diverges to infinity as $M \to \infty$.

Assume for A decay $p_1 = 0$:

$$-i\mathcal{M} = \frac{g^3/(2\pi)^4}{[(p_2+p_3)^2 - m_A^2c^2]} \int \frac{d^4q}{[q^2 - m_B^2c^2][q^2 - m_C^2c^2]}$$

Insert the $-M^2c^2/q^2-M^2c^2$ term into the integral and make substitutions $A \equiv Mc$, $B \equiv m_Bc$, + $C \equiv m_Cc$:

$$-i\mathcal{M} = \frac{g^3}{[(p_2+p_3)^2 - m_A^2c^2]} \int \frac{d^4q}{(2\pi)^4} \frac{-A^2}{(q^2-A^2)(q^2-B^2)(q^2-C^2)}$$

We evaluate only the integral and make use of two identities from J.J. Sakurai's <u>Advanced Quantum Mechanics</u>;

1) $$\frac{1}{abc} = 2\int_0^1 dx \int_0^{1-x} dy \frac{1}{[a+(b-a)x+(c-a)y]^3}$$

and 2) $$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+s-i\epsilon)^3} = \frac{i}{32\pi^2 s}.$$

We see that,

$$-A^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2-A^2)(q^2-B^2)(q^2-C^2)} = -2A^2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^4q}{(2\pi)^4} \frac{1}{[q^2-A^2+(A^2-B^2)x+(A^2-C^2)y]^3}$$

$$= -2A^2 \int_0^1 dx \int_0^{1-x} dy \frac{i}{32\pi^2} \frac{1}{[(A^2-B^2)x+(A^2-C^2)y-A^2+i\epsilon]} \qquad \epsilon \to 0$$

$$= \frac{-iA^2}{(4\pi)^2} \int_0^1 dx \left. \frac{\ln[(A^2-B^2)x+(A^2-C^2)y-A^2]}{A^2-C^2} \right|_0^{1-x}$$

$$= \frac{-iA^2}{(4\pi)^2} \int_0^1 dx \left\{ \frac{\ln[(C^2-B^2)x-C^2] - \ln[(A^2-B^2)x-A^2]}{A^2-C^2} \right\}$$

$$= \frac{-iA^2}{(4\pi)^2(A^2-C^2)} \left\{ \left(\frac{[(C^2-B^2)x-C^2]\ln[(C^2-B^2)x-C^2]}{C^2-B^2} - x \right)\Bigg|_0^1 - \left(\frac{[(A^2-B^2)x-A^2]\ln[(A^2-B^2)x-A^2]}{A^2-B^2} - x \right)\Bigg|_0^1 \right\}$$

$$= \frac{iA^2}{(4\pi)^2(A^2-C^2)}\left\{\left(\frac{-B^2\ln(-B^2)-(C^2-B^2)+C^2\ln(-C^2)}{C^2-B^2}\right) - \left(\frac{-B^2\ln(-B^2)-(A^2-B^2)+A^2\ln(-A^2)}{A^2-B^2}\right)\right\}$$

$$= \frac{iA^2}{(4\pi)^2(A^2-C^2)}\left\{\left(\frac{C^2\ln C^2-B^2\ln B^2-(C^2-B^2)+i\pi(C^2-B^2)}{C^2-B^2}\right) - \left(\frac{A^2\ln A^2-B^2\ln B^2-(A^2-B^2)+i\pi(A^2-B^2)}{A^2-B^2}\right)\right\}$$

$$= \frac{iA^2}{(4\pi)^2(A^2-C^2)}\left\{\frac{(C^2\ln C^2-B^2\ln B^2)}{C^2-B^2} - \frac{(A^2\ln A^2-B^2\ln B^2)}{A^2-B^2}\right\}$$

which as $A \to \infty$ gives a finite term,

$$\frac{i}{(4\pi)^2}\frac{(C^2\ln C^2-B^2\ln B^2)}{C^2-B^2}$$

and divergent term

$$-\frac{i}{(4\pi)^2}\ln A^2$$