THE FEYNMAN RULES

I. Introduction:

Very generally, interactions can be viewed as the result of a "scattering" operator (S), a unitary operator acting on an initial state ($\backslash i >$) leading to a multitude of possible final states such that:

$$S \ \backslash i >= \sum_{possible \ f} < f \mid S \mid i > . \ \backslash f >$$

The transition probability to a particular final state ($\backslash f >$) is given by the square of the **S matrix element** ($\mid S_{fi} \mid^2$) such that:

$$\boxed{S_{fi} =< f \mid S \mid i >}$$

The initial state ($\backslash i >$) must result in some combination of final states ($\backslash f >$), which may include ($\backslash i >$) itself in which case no interaction took place. To separate the non-interaction case from the cases with interaction, we introduce a transition operator **T** such that:

$$\boxed{S = I + iT}$$

With $T_{fi} =< f \mid S \mid i >$ when ($f \neq i$)

It is also convenient to redefine the operator (T) in a way that makes the energy momentum conservation explicit:

$$\boxed{T_{fi} = (2\pi)^4 \delta^4 (p_f - p_i) M_{fi}}$$

Where (p_i and p_f) are the initial and final four vector momentum respectively. (M_{fi}) also represented by (M) is known as amplitude.

II. Rules for Feynman diagrams:

The Feynman rules, are the set of instructions used to calculate the contribution of a particular diagram to the total amplitude (M) of the process.

Example:

The figure below illustrates the second order (two vertices) and forth order (four vertices) Feynman diagrams for the electron-electron (Moller) scattering. Each Feynman diagram corresponds to the probability amplitude for the process depicted in the diagram. In the feynman calculation Each diagram is contributing to the total probability amplitude of the process ($e + e \rightarrow e + e$).

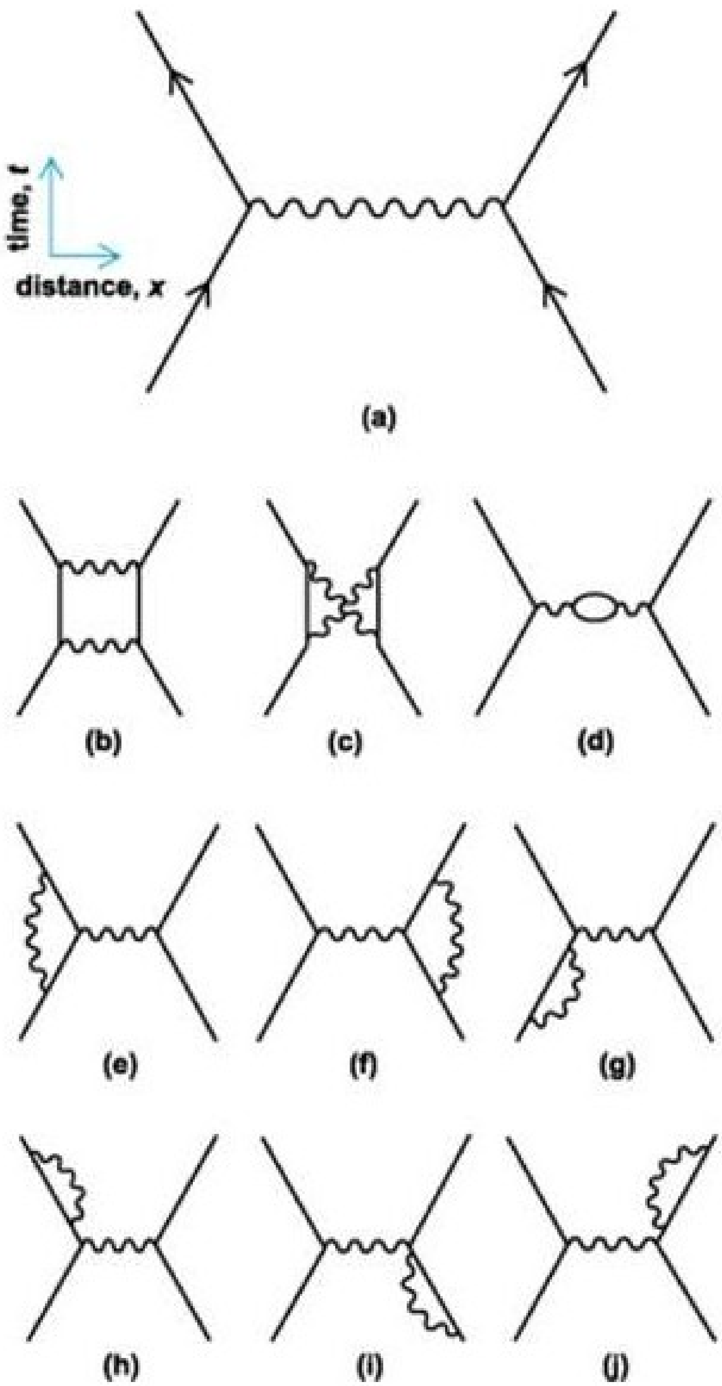


Figure 1: Feynman diagrams for electron-electron (Moller) scattering: (a) second-order diagram (two vertices); (b-j) fourth-order diagrams.

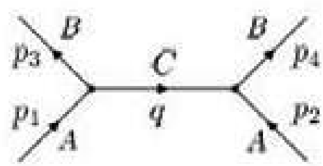
II-1. Feynman rules for a toy theory:

Consider a particular Feynman diagram, describing the interaction among spinless particles. To find the amplitude one needs to proceed as follow:

1. Label the incoming and outgoing four-momenta. Also label the internal momenta. Put an arrow on each line to keep track of the positive direction (if the arrow leads toward the vertex).

   Example:

   For the scattering: $A + A \rightarrow B + B$. One of the lowest Feynman diagrams corresponding to this interaction is:

2. For each vertex, write down a factor of $(-i\ g)$, where g is the coupling constant, it specifies the strength of the interaction.

3. For each internal line, write a factor $(\frac{i}{q_j^2 - m_j^2 c^2})$.

   Where (q_j) is the four vector momentum of the j^{th} internal line, and (m_j) is the mass of the particle the line describes. Notice that $(q_j^2 - m_j^2 c^2 \neq 0)$, because a virtual particle does not lie on its mass shell.

4. For each vertex, write a delta function of the form $[(2\pi)^4\ \delta^4(k_1 + k_2 + k_3)]$, to enforce conservation of energy and momentum. Where the k's are the four-momenta coming into the vertex.

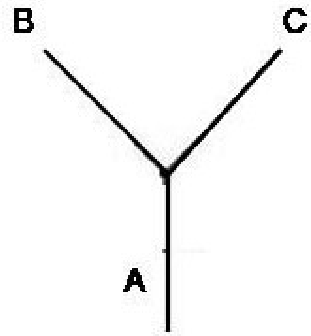
5. For each internal line (j), write down a factor $(\frac{1}{(2\pi)^4} d^4 q_j)$, and integrate over the internal momenta.

6. The result after enforcing overall conservation of energy and momentum is equal to $(-iM)$.

II-2. Application:

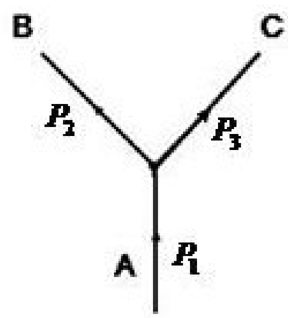
Imagine we have three spinless particles labeled A, B, and C with masses m_A, m_B, and m_C, each of which is its own antiparticle, with particle A being the heaviest of the three and in fact weighs more than B and C combined, so it can decay to $B+C$.

The lowest order Feynman diagram describing this decay ($A \to B + C$):



As can be seen there are no internal lines, and the diagram includes only a single vertex. Using the Feynman rules as listed in the previous paragraph:

(a) We label the incoming and outgoing four-momenta



(b) For the only vertex, we write down a factor of $(-i\ g)$.

(c) We skip step #3 since there are no internal lines

(d) For the only each vertex, we write a delta function $(2\pi)^4\ \delta^4(P_1 - P_2 - P_3)$ to enforce conservation of energy and momentum.

The result up to this step is:

$$(-ig).\ (2\pi)^4\ \delta^4(P_1 - P_2 - P_3)$$

(e) Enforcing overall conservation of energy and momentum ($P_1 = P_2 + P_3$)

The amplitude of the decay to the first order is:

$$\boxed{iM = ig \ or \ M = g}$$

Now assuming that the contribution of the higher order diagrams is negligible compared to the first order amplitude, the total amplitude is approximately equal to M. Next Using equations (6.31) and (6.32) from the book with:

1. S=1, (no groups of identical particles present in the final state of the decay)

2. $m_1 = m_A$, $m_2 = m_B$ and $m_3 = m_C$

The decay rate is then:

$$\Gamma = \frac{S \mid P \mid}{8\pi\hbar m_A^2 c} \mid M \mid^2 = \frac{\mid P \mid g^2}{8\pi\hbar m_A^2 c}$$

Where

$$\mid p \mid = \frac{c}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2}$$

And the life time of the particle A is

$$\tau = \frac{1}{\Gamma} = \frac{8\pi\hbar m_A^2 c}{\mid P \mid g^2}$$

We can actually check for the consistency of the above result

$$Unit(\tau) = Unit(\frac{8\pi\hbar m_A^2 c}{\mid P \mid g^2}) = \frac{(Mev \ s) \ (Mev/c^2)^2 \ c}{(Mev/c^2 \ c)^2 \ (Mev/c)} = s$$