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6.2 The Golden Rule

We're interested in decay rates and scattering crossections. Both of these are given from the transition rate.

$$\overset{W}{\cancel{\text{Transition rate}}} = \frac{2\pi}{\hbar} |M|^2 \times (\text{phase space})$$

The amplitude M is given from the Feynmann diagrams, while the phase space term is given from the masses, energies, + momenta of the particles involved. From a statistical mechanics point of view, this is the relative number of microstates, so the relative probability that the transition will occur. $|M|^2$ must be evaluated on a case-by-case basis, but the phase space term ~~is given by the mass~~ may be given fairly generally.

Decays

$1 \rightarrow 2 + 3 + \dots + n$

The differential decay rate is related simply to the transition rate:

$$d\Gamma = |M|^2 \cdot \frac{S}{2\hbar m_1} \cdot \prod_{i=2}^{n} \left(\frac{c\, d^3\vec{p}_i}{(2\pi)^3 \cdot 2E_i} \right) \cdot (2\pi)^4 \, \delta^4\!\left(p_1 - \sum_{i=2}^{n} p_i\right)$$

(Note that this "differential decay rate" is actually a differential, as opposed to the "differential cross section")

$$d\Gamma = \underbrace{|M|^2}_{\text{amplitude}} \cdot \underbrace{S}_{\substack{\text{adjust.}\\ \text{for ident.}\\ \text{particles}}} \underbrace{\frac{1}{2\hbar} \cdot \frac{1}{m_1} \prod_{i=2}^{n} \left(\frac{c\, d^3\vec{p}_i}{(2\pi)^3\, 2E_i} \right)}_{\text{phase space terms}} \cdot \underbrace{(2\pi)^4 \, \delta^4\!\left(p_1 - \sum_{i=2}^{n} p_i\right)}_{\text{conservation of energy}}$$

Usually interested in total decay rate, so we integrate over all momenta.

$$\Gamma = \frac{S}{\hbar m_1} \cdot \left(\frac{c}{16\pi^3}\right)^{n-1} \cdot (2\pi)^4 \cdot \frac{1}{2} \int |M|^2 \, \delta^4\!\left(p_1 - \sum_{i=2}^{n} p_i\right) \prod_{i=2}^{n} \left(\frac{1}{E_i}\, d^3\vec{p}_i \right)$$

For a decay into two particles:

$$\Gamma = \frac{S}{\hbar m_1} \left(\frac{c}{4\pi}\right)^2 \cdot \frac{1}{2} \int \frac{|M|^2}{E_2 E_3} \, \delta^4(p_1 - p_2 - p_3)\, d^3\vec{p}_2 \, d^3\vec{p}_3$$

~~Ex decay~~: A particle, mass m, decays into two massless particles, with amplitude $M(\vec{p}_2, \vec{p}_3)$. Find the decay rate.

$$\Gamma = \frac{S}{\hbar m_1}\left(\frac{c}{4\pi}\right)^2 \cdot \frac{1}{2}\int \frac{|M|^2}{E_2 E_3}\delta^4(p_1 - p_2 - p_3)\, d^3p_3\, d^3p_2$$

$$m_2 = m_3 = 0 \Rightarrow E_2 = |\vec{p}_2|c\,,\quad E_3 = |\vec{p}_3|c$$

$$\cancel{\Gamma = \frac{S}{\hbar m_1}\left(\frac{1}{4\pi}\right)^2 \cdot \frac{1}{2}\int \frac{|M|^2}{|\vec{p}_2||\vec{p}_3|}\delta^3(mc - \tfrac{E_2}{c} \ldots)}$$

Using $c = \hbar = 1$

$$\Gamma = \frac{S}{m_1}\left(\frac{1}{4\pi}\right)^2 \cdot \frac{1}{2}\int \frac{|M|^2}{|\vec{p}_2||\vec{p}_3|}\delta(m_1 - |\vec{p}_2| - |\vec{p}_3|)\,\delta^3(-\vec{p}_2 - \vec{p}_3)\, d^3\vec{p}_3\, d^3\vec{p}_2$$

$$= \frac{S}{m}\left(\frac{1}{4\pi}\right)^2 \cdot \frac{1}{2}\int \frac{|M|^2}{|\vec{p}_2||-\vec{p}_2|}\delta(m - |\vec{p}_2| - |\vec{p}_2|)\, d^3\vec{p}_2$$

$$= \frac{S}{m}\left(\frac{1}{4\pi}\right)^2 \cdot \frac{1}{2}\int \frac{|M|^2}{|\vec{p}_2|^2}\delta(m - 2|\vec{p}_2|)\, d^3\vec{p}_2 = \frac{S}{m}\left(\frac{1}{4\pi}\right)^2 \cdot \frac{1}{2}\int_0^\infty \frac{|M|^2}{|\vec{p}_2|^2}\delta(m - 2|\vec{p}_2|)\cdot 4\pi|\vec{p}_2|^2\, d|\vec{p}_2|$$

$$= \frac{S}{8\pi m}\int_0^\infty |M|^2 \cdot \frac{1}{2}\delta\left(|\vec{p}_2| - \frac{m}{2}\right) d|\vec{p}_2| \overset{\frac{m}{2} > 0}{=} \boxed{\frac{S}{16\pi m}\cdot |M|^2}$$

Putting c + $\hbar$ in: $\boxed{\Gamma = \frac{S}{16\pi\hbar m}|M|^2}$

Ex 2 decay: m_2 + m_3 given: $\Gamma = \frac{S}{2(4\pi)^2 m_1}\int \frac{|M|^2}{E_2 E_3}\delta(m_1 - E_2 - E_3)\,\delta^3(-\vec{p}_2 - \vec{p}_3)\, d^3\vec{p}_3\, d^3\vec{p}_2$ where $E_2 = \sqrt{m_2^2 + |p_2|^2}$, $E_3 = \sqrt{m_3^2 + |p_3|^2}$

$$= \frac{S}{2(4\pi)^2 m_1}\int \frac{|M|^2}{\sqrt{m_2^2 + |\vec{p}_2|^2}\sqrt{m_3^2 + |\vec{p}_3|^2}}\delta\left(m_1 - \sqrt{m_2^2 + \vec{p}_2^{\,2}} - \sqrt{m_3^2 + \vec{p}_3^{\,2}}\right) d^3\vec{p}_2$$

$$= \frac{S}{8\pi m_1}\int_0^\infty \frac{|M|^2}{\sqrt{m_2^2 + p^2}\sqrt{m_3^2 + p^2}}\delta\left(m_1 - \sqrt{m_2^2 + p^2} - \sqrt{m_3^2 + p^2}\right) d p$$

$$E \equiv \sqrt{m_2^2 + p^2} + \sqrt{m_3^2 + p^2}$$

$$\Rightarrow dE = \frac{E p}{\sqrt{m_2^2 + p^2}\sqrt{m_3^2 + p^2}}\, dp$$

$$\Rightarrow \Gamma = \frac{S}{8\pi m_1}\int_{m_2 + m_3}^\infty \frac{|M|^2 p}{E}\delta(m_1 - E)\, dE = \boxed{\frac{S p_0}{8\pi m_1^2}|M|^2} \overset{c\hbar}{=} \boxed{\frac{S p_0}{8\pi\hbar m_1^2 c}|M|^2}$$

if $m_1 > m_2 + m_3$

where $p_0 = p$ s.t. $E = m_1$

Reduces to $\boxed{\Gamma = \frac{S|\vec{p}|}{8\pi\hbar m_1^2 c}|M|^2}$

Scattering

Say two particles collide: $1+2 \to 3+\ldots+n$

The cross section is also related simply to the transition rate:

$$d\sigma = |M|^2 \cdot S \cdot \frac{\hbar^2}{4\sqrt{(p_1\cdot p_2)^2-(m_1 m_2 c^2)^2}} \prod_{i=3}^{n}\left(\frac{c\, d^3\vec{p}_i}{(2\pi)^3 \cdot 2E_i}\right)\cdot (2\pi)^4 \delta^4\Big(p_1+p_2-\sum_{i=1}^{3} p_i\Big)$$

$\frac{d\sigma}{d\Omega}$ = diff. cross section for scattering of particle 3 into $d\Omega = \int d\sigma \, d^3\vec{p}_n \cdots d^3\vec{p}_4 \cdot d|\vec{p}_3|$

Ex: 2 body scattering $\quad 1+2\to 3+4 \qquad$ CM frame $\Rightarrow \vec{p}_2 = -\vec{p}_1 \Rightarrow p_1\cdot p_2 = E_1E_2 - \vec{p}_1^{\,2}$

$$\Rightarrow \sqrt{(p_1\cdot p_2)^2-(m_1m_2)^2} = \sqrt{(E_1E_2-\vec{p}_1^{\,2})^2-(m_1m_2)^2} = \sqrt{(E_1E_2-\vec{p}_1^{\,2})^2-(E_1^2-|\vec{p}_1|^2)(E_2^2-|\vec{p}_2|^2)}$$

$$= \sqrt{-2E_1E_2\vec{p}_1^{\,2}+E_1^2\vec{p}_1^{\,2}+E_2^2\vec{p}_1^{\,2}} = |E_1+E_2|\,|\vec{p}_1|$$

$$d\sigma = \left(\frac{1}{8\pi}\right)^2 \frac{S|M|^2}{(E_1+E_2)|\vec{p}_1|}\,\frac{d^3\vec{p}_3\, d^3\vec{p}_4}{E_3E_4}\,\delta^4(p_1+p_2-p_3-p_4)$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi}\right)^2 \int \frac{S|M|^2}{(E_1+E_2)|\vec{p}_1|}\,\frac{1}{\sqrt{\vec{p}_3^{\,2}+m_3^2}\sqrt{\vec{p}_4^{\,2}+m_4^2}}\,\delta\Big(E_1+E_2-\sqrt{m_3^2+\vec{p}_3^{\,2}}-\sqrt{m_4^2+\vec{p}_4^{\,2}}\Big)\,\delta^3(-\vec{p}_3-\vec{p}_4)\,d^3\vec{p}_4\, d|\vec{p}_3|$$

$$= \left(\frac{1}{8\pi}\right)^2 \int_0^\infty \frac{S|M|^2}{(E_1+E_2)|\vec{p}_1|}\,\frac{1}{\sqrt{m_3^2+p^2}\sqrt{m_4^2+p^2}}\cdot\delta\Big(E_1+E_2-\sqrt{m_3^2+p^2}-\sqrt{m_4^2+p^2}\Big)\,dp$$

$$= \frac{S}{(8\pi)^2|\vec{p}_1|(E_1+E_2)}\cdot\frac{|M|^2 p_0}{(E_1+E_2)} = \left(\frac{1}{8\pi}\right)^2\frac{S|M|^2}{(E_1+E_2)^2}\,\frac{|\vec{p}_3|}{|\vec{p}_1|} = \boxed{\left(\frac{1}{8\pi}\right)^2\frac{S|M|^2}{(E_1+E_2)^2}\,\frac{|\vec{p}_f|}{|\vec{p}_i|}}$$