Outline: Lifetimes and Cross Sections.

- Importance: Original conditions → Feynman calculus → Interactions.
- Lifetime: Decays & Branching ratios.
- Scattering: Cross section, Rutherford scattering, QM approach, Yukawa.

Importance:

- Once we have information about the interacting particles, it is possible to perform a Feynman calculation for an in-depth study of the type of interaction. One can also know the outcome of the given interaction.
- Many experiments in high-energy and particle physics involve collisions of particles. By studying parameters, like the cross section, it is possible to make predictions about the results and prepare a set of initial conditions.
- The main interest in central forces arose out of the study of planetary motions. But its applications in particle physics are vasts, as the scattering by a central force plays an important role in particle interaction. The classical treatment of the collisions is not always a good approximation, so it is necessary to introduce some QM.
- Lifetimes end up being the time that it takes a certain particle to decay. However, the decay products are not always the same. Hence the relation between lifetimes gives the probability for a certain decay to take place.

Lifetimes:

- Since not every decay has the same lifetime, it's more appropriate to calculate the average lifetime of all the particles in a large sample.
- Given a large sample of one type of particles, they all have the same probability of decaying in an arbitrary period of time.

$$dN(t) = -\Gamma N(t)\, dt \quad ; \text{ where: } \Gamma = \text{decay parameter for initial conditions.}$$

$$\int_0^N \frac{dN(t)}{N(t)} = -\int_0^t \Gamma\, dt \Rightarrow \ln\left(\frac{N(t)}{N_0}\right) = -\Gamma t$$

$$\boxed{N(t) = N_0 e^{-\Gamma t}}$$

- As it can be seen the number of particle decays exponentially with time. It can be proven that Γ is related to lifetime (τ). Assuming that the probability is normalized, by a factor k:

$$1 = \int_0^\infty k N(t)\, dt \Rightarrow 1 = \int_0^\infty k N_0 e^{-\Gamma t}\, dt$$

$$1 = k N_0 \int_0^\infty e^{-\Gamma t}\, dt = k N_0 \left[-\frac{1}{\Gamma}\left(e^{-\infty} - e^{0}\right)\right]$$

$$1 = \frac{k N_0}{\Gamma} \Rightarrow \boxed{k = \frac{\Gamma}{N_0}}$$

Now, taking the expected value of the distribution, one gets:

$$\langle t \rangle = \tau = \int_0^\infty k t N(t)\, dt = \int_0^\infty k t N_0 e^{-\Gamma t}\, dt$$

$$\tau = kN_0 \int_0^{\infty} t e^{-\Gamma t} dt \quad ; \quad \int_0^{\infty} t e^{-\Gamma t} dt = t\left(\frac{e^{-\Gamma t}}{-\Gamma}\right)\Bigg|_0^{\infty} - \left(\frac{e^{-\Gamma t}}{-\Gamma^2}\right)\Bigg|_0^{\infty}$$

$$\int_0^{\infty} t e^{-\Gamma t} dt = \frac{1}{\Gamma^2}$$

Then:

$$\tau = \frac{kN_0}{\Gamma^2} \quad ; \quad \text{but} : kN_0 = \Gamma$$

So: $\boxed{\tau = \frac{1}{\Gamma}}$

Now if one assumes that a particle can decay into different products, each decay with certain intrinsic properties, then there should be a decay rate for each decay. Hence one would expect:

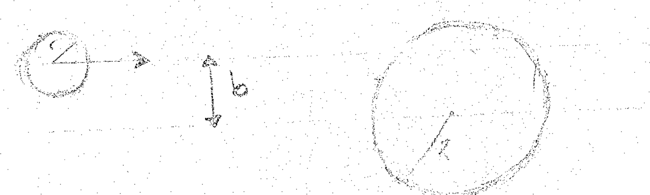
$$\boxed{\Gamma_{tot} = \sum_{i=1}^{n} \Gamma_i} \quad , \text{ where: } \Gamma_{tot} = \frac{1}{\tau}$$

Finally, thinking about the probability of each decay actually happening, a relation with Γ instantly comes up. Being the branching ratio for the i-th decay:

$$\boxed{Br_i = \frac{\Gamma_i}{\Gamma_{tot}}}$$

Cross Sections:

- The cross section is basically reduced to the effective area of impact, for a given collision. As particles will get scattered while colliding, one can define this cross section as a ratio between the amount of incident particles and those that get scattered into a solid angle $d\Omega$. It first can be seen as two spheres, comparable in size, colliding, for practical purposes:



'b' will known as the parameter of impact

Since the colliding particles are being considered hard spheres, one can generalize this situation to a beam of incident particles with b and $b+db$ as impact parameter, being scattered with an angle between θ and $\theta+d\theta$

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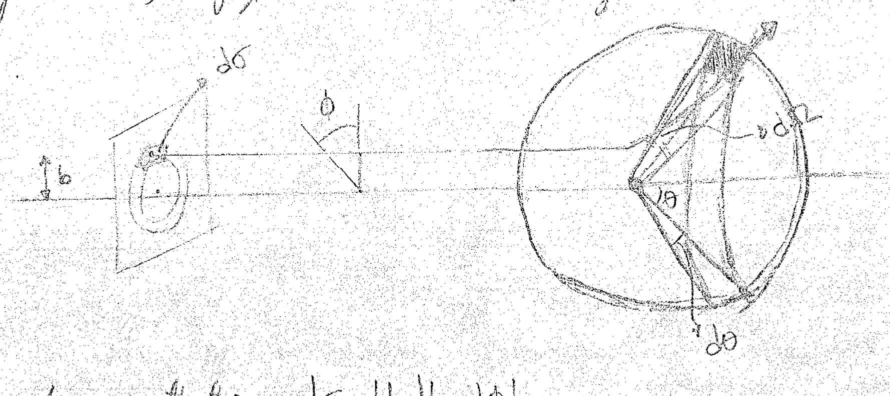


It can be seen that: $d\sigma = |b\,db \cdot d\phi|$

And remembering: $d\Omega = |\sin\theta \cdot d\theta \cdot d\phi|$

$$D(\theta)\, d\Omega = \frac{\#\text{ scattered particles}}{\#\text{ incident particles}} = d\sigma$$

$$d\sigma = D(\theta)\, d\Omega$$

$$D(\theta) = \frac{d\sigma}{d\Omega} \Rightarrow D(\theta) = \left| \frac{b\, db\, d\phi}{\sin\theta\, d\theta\, d\phi} \right|$$

$$\boxed{D(\theta) = \left| \frac{b}{\sin\theta} \cdot \frac{db}{d\theta} \right|}$$

As expected, this differential scattering cross section is independent of ϕ; and it was expected as many of the scattering potentials, produced by the charged particle, are spherically symmetrical.

- Rutherford scattering considers the Coulomb potential as the scattering potential. Being q_1 and q_2 the interacting charged particles, the central force between them is:

$$F = \frac{q_1 \cdot q_2}{r^2}, \quad \text{Coulomb's Force.}$$

As F obbeys to the inverse-square law force, because it's a central force, it can be written as:

$$F(r) = -\frac{k}{r^2} \quad , \text{ with: } k = -q_1 q_2 \text{ in this particular case.}$$

Then the potential will be: for $k>0$

$$V(r) = \int F(r)\, dr \Rightarrow V(r) = -\frac{k}{r}$$

Now that the force is given, the orbit of the particle can be predicted:

$$\eta = \eta' - \int \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{2mku}{l^2} - u^2}}$$

Where: $u = \frac{1}{r}$

E = energy of particle

l = angular momentum

m = mass of particle

This integral can be associated with the primitive:

$$\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}} = \frac{1}{\sqrt{-\gamma}} \,\text{arc.cos}\left(-\frac{\beta + 2\gamma x}{\sqrt{q}}\right), \quad q = \beta^2 - 4\alpha\gamma$$

Making: $\alpha = \frac{2mE}{l^2}$, $\beta = \frac{2mk}{l^2}$, $\gamma = -1$

$$\int \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{2mku}{l^2} - u^2}} = \text{arc cos}\left(-\frac{\frac{2mk}{l^2} - 2u}{\sqrt{\frac{4m^2k^2}{l^4} + \frac{8mE}{l^2}}}\right)$$

$$\sqrt{\frac{4m^2k^2}{l^4} + \frac{8mE}{l^2}} = \sqrt{\frac{4m^2k^2}{l^4}\left(1 + \frac{2El^2}{mk^2}\right)} = \frac{2mk}{l^2}\sqrt{1 + \frac{2El^2}{mk^2}}$$

$$\frac{2mk}{l^2} - 2u = \frac{2mk}{l^2}\left[1 - \frac{ul^2}{mk}\right] = -\frac{2mk}{l^2}\left[\frac{l^2u}{mk} - 1\right]$$

$$\int \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{2mku}{l^2} - u^2}} = \text{arc. cos}\left(\frac{-\frac{2mk}{l^2}\left[\frac{l^2u}{mk} - 1\right]}{\frac{2mk}{l^2}\sqrt{1 + \frac{2El^2}{mk^2}}}\right)$$

Then:

$$\eta = \eta' - \arccos\left(\frac{\frac{l^2 u}{mk} - 1}{\sqrt{1 + \frac{2El^2}{mk^2}}}\right)$$

Solving for $u = \frac{1}{r}$

$$\frac{l^2 u}{mk} - 1 = \sqrt{1 + \frac{2El^2}{mk^2}} \cos(\eta - \eta')$$

$$\boxed{\frac{1}{r} = \frac{mk}{l^2}\left[1 + \sqrt{1 + \frac{2El^2}{mk^2}} \cos(\eta - \eta')\right]}$$

The eccentricity of the orbit, ϵ, is given by:

$$\epsilon = \sqrt{1 + \frac{2El^2}{mk^2}}$$

Now is time for the initial conditions of the Rutherford scattering:

$$l = m v_0 \cdot b = b\sqrt{2mE} \quad , \quad v_0 = \text{incident speed.}$$

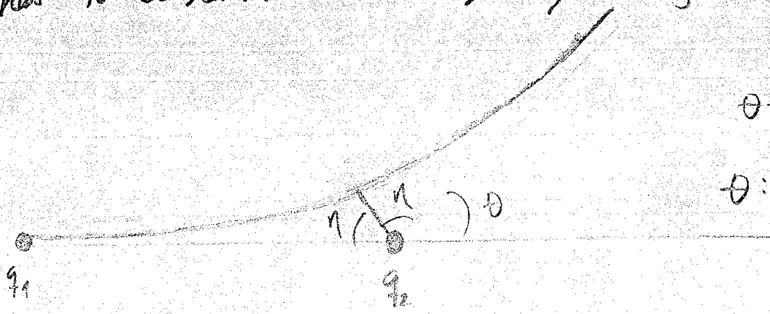
$E > 0 \rightarrow$ orbit is hyperbolic

Choosing the periapsis as the point to study - closest approach with $\eta' = \pi$ & $\eta = 0$:

$$\epsilon = \sqrt{1 + \left(\frac{2Eb}{q_1 q_2}\right)^2}$$

$$\frac{1}{r} = \frac{m q_1 q_2}{l^2}(\epsilon \cos\eta - 1)$$

It has to be remembered that the scattering is:



$\theta + 2\eta = \pi$

θ: scattering angle.

The orbit of the incoming asymptote is determined by $r \to \infty$, then.

$$\frac{1}{r}^{\to 0} = \frac{m q_1 q_2}{l^2}(\epsilon \cos\eta - 1) \Rightarrow \epsilon \cos\eta = 1$$

$$\boxed{\cos\eta = \frac{1}{\epsilon}}$$

but since: $\eta = \frac{\theta - \pi}{2}$

$$\sin\left(\frac{\theta}{2}\right) = \frac{1}{\epsilon} \qquad \Rightarrow \epsilon = \operatorname{cosec}\left(\frac{\theta}{2}\right)$$

$$\operatorname{cotg}^2\left(\frac{\theta}{2}\right) = \operatorname{cosec}^2\left(\frac{\theta}{2}\right) - 1 \quad \Rightarrow \boxed{\operatorname{cotg}^2\left(\frac{\theta}{2}\right) = \epsilon^2 - 1}$$

Substituting ϵ:

$$\operatorname{cotg}^2\left(\frac{\theta}{2}\right) = \cancel{1+}\left(\frac{2Eb}{q_1 q_2}\right)\cancel{-1} \Rightarrow \operatorname{cotg}\left(\frac{\theta}{2}\right) = \frac{2Eb}{q_1 q_2}$$

$$\boxed{b = \frac{q_1 \cdot q_2 \operatorname{cotg}(\theta/2)}{2E}}$$

D(θ) can now be calculated:

$$\frac{db}{d\theta} = \frac{q_1 q_2}{2E}\left(-\frac{1}{\sin^2(\theta/2)}\right)$$

$$D(\theta) = \left| \frac{q_1 q_2}{2\sin(\theta/2)E} \cot(\theta/2) \left[\frac{q_1 q_2}{2E}\left(-\frac{1}{\sin^2(\theta/2)}\right)\right] \right|$$

$$\boxed{D(\theta) = \left(\frac{q_1 q_2}{4E \sin^2(\theta/2)} \right)^2}$$

Finally, the total cross section σ can be calculated:

$$\sigma = \int_0^\pi D(\theta)\, d\Omega = \left(\frac{q_1 q_2}{4E}\right)^2 \int_0^{2\pi} d\phi \int_0^\pi \frac{\sin\theta\, d\theta}{\sin^4(\theta/2)}$$

$$\boxed{\sigma = 2\pi \left(\frac{q_1 q_2}{4E}\right)^2 \int_0^\pi \frac{\sin\theta\, d\theta}{\sin^4(\theta/2)} = \infty}$$

This result is not surprising at all. It is important to remember that the Coulomb potential has very long range; its effects ex tend to infinity. Hence all the particles in an incident beam will be, at some point, scattered by this long-ranged potential.

Quantum Mechanical treatment:

- For simplicity purposes, instead of considering to particles, there will be a reduction of the 2-body problem, in which it will considered the scattering one body of reduced mass μ, by a potential $V(r_1 - r_2) = V(r)$

- Since this a QM approach, the right way to represent the interaction is with the wave function. And considering a stationary scattering state, the wave function representing such state will be the superposition of the incoming particle wave function (e^{ikz}) and the scattered particle wave function. Hence:

$$\boxed{\psi_k(\vec{r}) \sim e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r}}$$; where: $\frac{e^{ikr}}{r}$ = radial dependence

$f(\theta,\phi)$ = scattering amplitude

- Again, for simplicity, instead of calculate using beams of particles, one considers probability fluids in this steady state, for the incident and scattered currents. For a wave function $\varphi(\vec{r})$, this probability is:

$$\vec{J}(\vec{r}) = \frac{1}{\mu} Re\left[\varphi^*(\vec{r})\frac{\hbar}{i}\nabla\varphi(\vec{r})\right]$$

$\rightarrow \varphi(r) = e^{ikz}$ for the incident particle, so:

$$\vec{J}(r)_i = \frac{1}{\mu} Re\left[e^{-ikz}\frac{\hbar}{i}(ik)e^{ikz}\right]$$

$$\boxed{\vec{J}(r)_i = \frac{\hbar k}{\mu}}$$

Since the scattered wave function depends of spherical coordinates, ∇ Should be in spherical coordinates too. For our scattered wave function:

$$\psi_{(r)_s} = f(\theta,\phi)\frac{e^{ikr}}{r} \quad , \quad (\nabla)_r = \frac{\partial}{\partial r}$$

$$\vec{J}_{(r)_r} = \frac{1}{\mu}\, Re\left[f(\theta,\phi)\frac{e^{-ikr}}{r}\cdot\frac{\hbar}{i}\cdot f(\theta,\phi)\cdot\left(\frac{ik}{r}e^{ikr} - \frac{e^{ikr}}{r^2}\right)\right]$$

$$\boxed{\vec{J}_{(r)_r} = \frac{\hbar k}{\mu}\cdot\frac{1}{r^2}\left|f(\theta,\phi)\right|^2}$$

If the stationary state is $\psi_{(r)}$, then the incident flux (F_i) must be proportional of the vector $J_{(r)i}$:

$$F_i = C\cdot\left|\vec{J}_{(r)i}\right| \Rightarrow F_i = C\cdot\frac{\hbar k}{\mu} \quad , \text{ where : } C = \text{proportionality constant}$$

In analogy, the number dn, of particles that reach the detector per unit time, is proportional to the spherical flux vector J_d across a surface ds. Hence:

$$dn = C\ \vec{J_d}\cdot\vec{ds} \quad , \text{ but: } \vec{J_d}\cdot\vec{ds} = \vec{J}_{(r)_r}\, r^2 d\Omega$$

$$\boxed{dn = C\cdot\frac{\hbar k}{\mu}\left|f(\theta,\phi)\right|^2 d\Omega}$$

Comparing to the classical result:

$$\boxed{D(\theta,\phi) = \left|f(\theta,\phi)\right|^2} \rightsquigarrow$$ This introduces the possibility of spherical asymmetry (dependance of ϕ)

If one were to construct step by step the stationary wave function, one would get what is known as the Born Expansion. And taking into account the first order of the potential, the result would be the Born Approximation. So, for an incident wave function $e^{i\vec{k}\cdot\vec{r}}$, and a potential $U(\vec{r}) = \frac{2\mu}{\hbar^2} V(\vec{r})$:

$$\boxed{f^{(B)}(\theta,\phi) = -\frac{1}{4\pi} \int d^3r' \, e^{-i\vec{k}\cdot\vec{r}'} U(\vec{r}')}$$ ⇝ simply put, the scattering amplitude is the Fourier Transform of the Potential.

The accuracy of this approximation can be studied by using the Yukawa potential as the scattering potential:

$$V(\vec{r}) = V_0 \frac{e^{-\alpha r}}{r} \quad ; \quad V_0, \alpha \in \mathbb{R} \quad , \alpha > 0$$

Substituting $V(r)$ in $f^{(B)}(\theta,\phi)$:

$$f^{(B)}(\theta,\phi) = -\frac{1}{4\pi} \frac{2\mu}{\hbar^2} V_0 \int d^3r' \, e^{i\vec{k}\cdot\vec{r}'} \frac{e^{-\alpha r}}{r}$$

since the potential only depends on $\vec{r}$, angular integrations are easier:

$$f^{(B)}(\theta,\phi) = -\frac{1}{\cancel{4\pi}} \frac{2\mu V_0}{\hbar^2} \frac{\cancel{4\pi}}{\vec{k}} \int_0^\infty r \, dr \sin(|\vec{k}| r) \frac{e^{-\alpha r}}{\cancel{r}}$$

$$\boxed{f^{(B)}(\theta,\phi) = -\frac{2\mu V_0}{\hbar^2} \cdot \frac{1}{\alpha^2 + |\vec{k}|^2}}$$; it can be seen that: $|\vec{k}| = 2k \sin(\theta/2)$

Then: $$\boxed{D^{(B)}(\theta,\phi) = \frac{4\mu^2 V_0^2}{\hbar^4} \cdot \frac{1}{\left[\alpha^2 + 4k^2 \sin^2(\theta/2)\right]^2}}$$

One would expect that the Yukawa potential approaches the Coulomb potential as $\alpha = 0$. Then:

$$V(r) = V_0 \frac{e^{-(0)r}}{r} \Rightarrow V(r) = \frac{V_0}{r} \quad , \text{ where } V_0 = q_1 \cdot q_2$$

So:

$$f^{(B)}_{(\theta,\phi)} = -\frac{2\mu\, q_1 \cdot q_2}{\hbar^2} \cdot \frac{1}{2k \sin^2(\theta/2)}$$

$$f^{(B)}_{(\theta,\phi)} = -\frac{q_1 \cdot q_2\, \mu}{\hbar^2} \cdot \frac{1}{2k \sin^2(\theta/2)}$$

or what's more:

$$D^{(B)}_{(\theta,\phi)} = \frac{4\mu^2}{\hbar^4} \frac{q_1^2 \cdot q_2^2}{16 k^4 \sin^4(\theta/2)} \quad ; \text{ with: } \quad E_k = \frac{\hbar^2 k^2}{2\mu}$$

$$\boxed{D^{(B)}_{(\theta,\phi)} = \frac{q_1^2 \cdot q_2^2}{16 E^2 \sin^4(\theta/2)}}$$