Shen's presentation
Sec. 6.2 (2nd half)

Golden Rule for Scattering

Suppose particles 1 and 2 collide, producing particles 3, 4, ⋯, n:

$$1 + 2 \longrightarrow 3 + 4 + \cdots + n$$

The cross section is given by the formula

$$d\sigma = |\mathcal{M}|^2 \frac{\hbar^2 S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \left[\frac{c\, d^3\vec{p}_3}{(2\pi)^3 2E_3} \frac{c\, d^3\vec{p}_4}{(2\pi)^3 2E_4} \cdots \frac{c\, d^3\vec{p}_n}{(2\pi)^3 2E_n}\right]$$

$$\times (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4 - \cdots - p_n)$$

Example. Two-Body Scattering in the CM Frame

Consider the process in the CM frame

$$1 + 2 \longrightarrow 3 + 4$$

If the amplitude is $\mathcal{M}$, calculate the differential cross section.

Solution: according to the formula

$$d\sigma = |\mathcal{M}|^2 \frac{\hbar^2 S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \left(\frac{c\, d^3\vec{p}_3}{(2\pi)^3 2E_3} \frac{c\, d^3\vec{p}_4}{(2\pi)^3 2E_4}\right)$$

$$\times (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

$$= \frac{|\mathcal{M}|^2 \hbar^2 S c^2}{(8\pi)^2} \frac{1}{\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \frac{d^3\vec{p}_3\, d^3\vec{p}_4}{E_3 E_4} \delta^4(p_1 + p_2 - p_3 - p_4)$$

In the CM frame, $\vec{P}_2 = -\vec{P}_1$

$$P_1 = \left(\frac{E_1}{c}, \vec{P}_1\right)$$

$$P_2 = \left(\frac{E_2}{c}, -\vec{P}_1\right)$$

$$\therefore \quad P_1 \cdot P_2 = \frac{E_1 E_2}{c^2} + \vec{P}_1^{\,2}$$

$$(P_1 \cdot P_2)^2 = \left(\frac{E_1 E_2}{c^2}\right)^2 + 2|\vec{P}_1|^2 \frac{E_1 E_2}{c^2} + |\vec{P}_1|^4$$

we also have

$$\begin{aligned}
(m_1 m_2 c^2)^2 &= m_1 c^2 \cdot m_2 c^2 \\
&= P_1^2 \, P_2^2 \\
&= \left[\left(\frac{E_1}{c}\right)^2 - |\vec{P}_1|^2\right]\left[\left(\frac{E_2}{c}\right)^2 - |\vec{P}_1|^2\right] \\
&= \left(\frac{E_1 E_2}{c^2}\right)^2 - |\vec{P}_1|^2\left(\frac{E_1}{c}\right)^2 - |\vec{P}_1|\left(\frac{E_2}{c}\right)^2 + |\vec{P}_1|^4
\end{aligned}$$

$$\begin{aligned}
\therefore \quad & (P_1 \cdot P_2)^2 - (m_1 m_2 c^2)^2 \\
&= \left(\frac{E_1 E_2}{c^2}\right)^2 + 2|\vec{P}_1|^2 \frac{E_1 E_2}{c^2} + |\vec{P}_1|^4 \\
&\quad - \left[\left(\frac{E_1 E_2}{c^2}\right)^2 - |\vec{P}_1|^2\left(\frac{E_1}{c}\right)^2 - |\vec{P}_1|\left(\frac{E_2}{c}\right)^2 + |\vec{P}_1|^4\right] \\
&= 2|\vec{P}_1|^2 \frac{E_1 E_2}{c^2} + |\vec{P}_1|^2\left(\frac{E_1}{c}\right)^2 + |\vec{P}_1|\left(\frac{E_2}{c}\right)^2 \\
&= \left(\frac{E_1}{c} + \frac{E_2}{c}\right)^2 |\vec{P}_1|^2
\end{aligned}$$

$$\therefore \quad \sqrt{(P_1 \cdot P_2)^2 - (m_1 m_2 c^2)^2} = \frac{(E_1 + E_2)|\vec{P}_1|}{c}$$

Substitute this in the expression of $d\sigma$, we have

$$d\sigma = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{|M|^2 S c}{(E_1 + E_2)|\vec{P}_1|} \frac{d^3\vec{P}_3 \, d^3\vec{P}_4}{E_3 E_4} \delta^4(P_1 + P_2 - P_3 - P_4)$$

we can write

$$\delta^4(P_1 + P_2 - P_3 - P_4) = \delta\left(\frac{E_1 + E_2 - E_3 - E_4}{c}\right)\delta^3(-\vec{P}_3 - \vec{P}_4)$$

Replace E_3 by $E_3 = \sqrt{|\vec{P}_3|^2 c^2 + m_3^2 c^4}$

E_4 by $E_4 = \sqrt{|\vec{P}_4|^2 c^2 + m_4^2 c^4}$

and carry out the integration over $\vec{P}_4$ (which sends $\vec{P}_4$ to $-\vec{P}_3$), we have

$$d\sigma = \left(\frac{\hbar}{8\pi}\right)^2 \frac{|M|^2 S c}{(E_1+E_2)|\vec{P}_1|} \frac{\delta\left(\frac{E_1+E_2}{c} - \sqrt{|\vec{P}_3|^2 + m_3^2c^2} - \sqrt{|\vec{P}_3|^2 + m_4^2c^2}\right)}{\sqrt{|\vec{P}_3|^2 + m_3^2c^2}\sqrt{|\vec{P}_3|^2 + m_4^2c^2}} d^3\vec{P}_3$$

Consider $d^3\vec{P}_3 = \rho^2 d\rho\, d\Omega$, where $\rho = |\vec{P}_3|$

we have

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1+E_2)|\vec{P}_1|}\int_0^\infty |M|^2 \frac{\delta\left(\frac{E_1+E_2}{c} - \sqrt{\rho^2+m_3^2c^2} - \sqrt{\rho^2+m_4^2c^2}\right)}{\sqrt{\rho^2+m_3^2c^2}\sqrt{\rho^2+m_4^2c^2}}\rho^2 d\rho$$

Let $E = c\left(\sqrt{\rho^2+m_3^2c^2} + \sqrt{\rho^2+m_4^2c^2}\right)$

Then
$$dE = c\left(\frac{\rho\, d\rho}{\sqrt{\rho^2+m_3^2c^2}} + \frac{\rho\, d\rho}{\sqrt{\rho^2+m_4^2c^2}}\right)$$

$$= \frac{c\left(\sqrt{\rho^2+m_4^2c^2} + \sqrt{\rho^2+m_3^2c^2}\right)}{\sqrt{\rho^2+m_3^2c^2}\sqrt{\rho^2+m_4^2c^2}}\rho\, d\rho$$

$$= \frac{E\rho\, d\rho}{\sqrt{\rho^2+m_3^2c^2}\sqrt{\rho^2+m_4^2c^2}}$$

$$\therefore \quad \frac{\rho^2 d\rho}{\sqrt{\rho^2+m_3^2c^2}\sqrt{\rho^2+m_4^2c^2}} = \frac{\rho}{E}dE$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1+E_2)|\vec{P}_i|} \int_0^\infty |M|^2 \frac{p}{E}\, \delta\left(\frac{E_1+E_2-E}{c}\right) dE$$

$$= \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc^2|M|^2}{(E_1+E_2)^2} \frac{|\vec{P}_f|}{|\vec{P}_i|}$$