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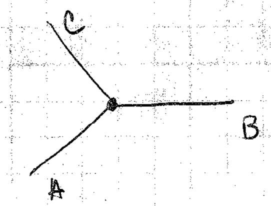
Higher-Order Feynman Diagrams

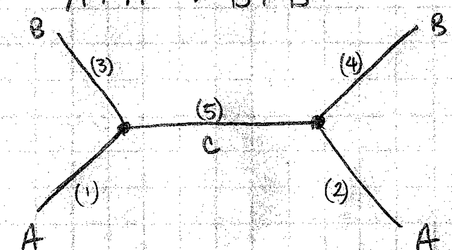
see Sakurai

Keeping in touch with our "toy theory," with the primitive vertex:



We want to consider more complicated processes, represented by higher-order Feynman diagrams. So far, we have considered only the lowest-order diagrams - Griffiths calls them "tree level" Feynman diagrams. Consider the case

$$A + A \to B + B$$



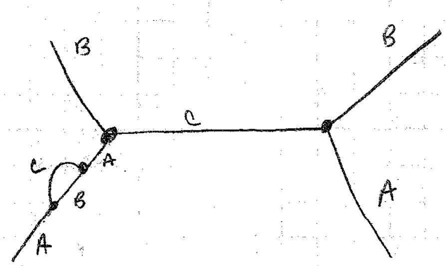
From Feynman's second rule, and from the analysis of scattering,

$$|\mathcal{M}| \propto g^2$$

there are 2 vertices

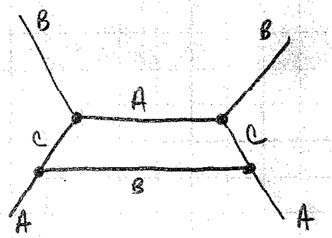
But, there are several diagrams in which $\underline{4}$ vertices contribute. For example, we can "add" lines to the diagram...

(I)



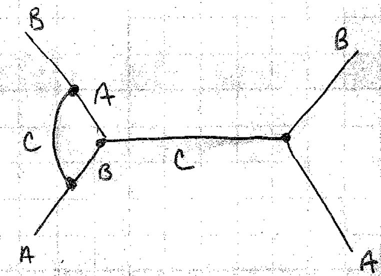
- Here, the added line starts on (1) and ends on (1).

(II)



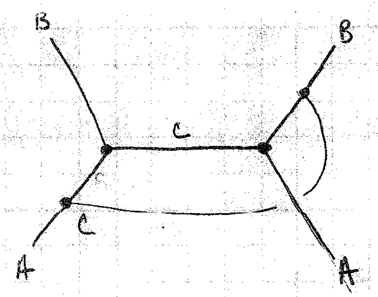
⇐ Here, the added line starts on (1) but ends on (2).

Added line starts on (1), ends on (3). ⇒



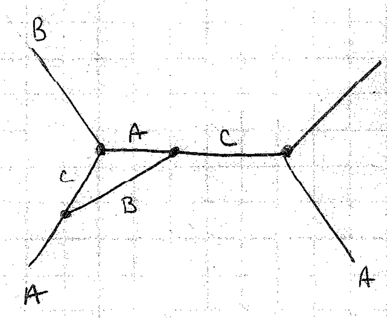
(III)

(IV)



⇐ Starts on (1), ends on (4)

Here, the added line starts on (1) and ends on (5). ⇒



(V)

So, there are 5 4^{th} order diagrams for the case that the added line originates on (1). Similarly, there are 5 for the case of adding a line from (2).

But, II is already one of these cases! So, there are $\underline{4}$ new 4^{th} order diagrams.

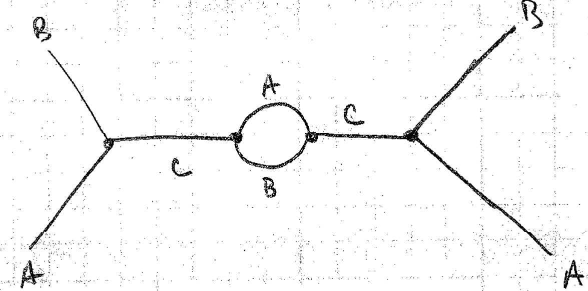
Continuing in this pattern, we see:

Originating from	Number of New Diagrams
(1)	5
(2)	4
(3)	3
(4)	2
(5)	1

Adding these, there are $\boxed{15}$ 4^{th} order diagrams for scattering $A+A \to B+B$

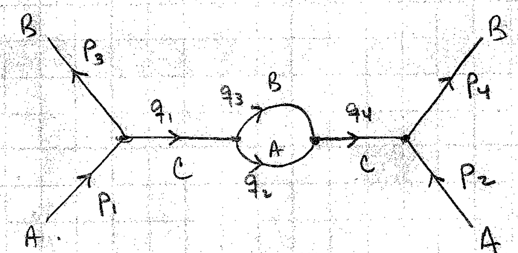
And, another 15 for the "twisted" version, where the outgoing B lines are crossed with one another.

To keep things simple, we will only consider the case of connecting (5) to itself:



So, let's follow Feynman's Rules & calculate the amplitude $\mathcal{M}$ for such a process:

1) Label all momenta:



∗ NB: draw arrows to keep track of "positive" direction. Also, recall external momenta are p_i and internal momenta are q_i.

2) This is a 4^{th} order diagram, and, for each vertex, there is a factor of $-ig$

$$\Rightarrow \quad (-ig)(-ig)(-ig)(-ig) = g^4$$

3) Now, since there are 4 internal lines, we write $\dfrac{i}{q_j^2 - m_j c^2}$, so,

$$\frac{i}{q_1^2 - m_1 c^2} \cdot \frac{i}{q_2^2 - m_2 c^2} \cdot \frac{i}{q_3^2 - m_3 c^2} \cdot \frac{i}{q_4^2 - m_4 c^2} = \frac{1}{(q_1^2 - m_1 c^2)(q_2^2 - m_2 c^2)(q_3^2 - m_3 c^2)(q_4^2 - m_4 c^2)}$$

4) Now, to conserve energy & momentum:

$$(2\pi)^{16}\, \delta^4(p_1 - q_1 - p_3)\, \delta^4(q_1 - q_3 - q_2)\, \delta^4(q_3 + q_2 - q_4)\, \delta^4(q_4 + p_2 - p_4)$$

Make sure to understand all the incoming / outgoing momentum sign conventions...

5) 4 internal lines, so, now,

$$\frac{1}{(2\pi)^{16}}\, d^4 q_1\, d^4 q_2\, d^4 q_3\, d^4 q_4$$

, and we will integrate over these momenta.

So, we now have:

$$-i\mathcal{M} = \int \frac{g^4 \delta^4(p_1-p_3-q_1)\delta^4(q_1-q_2-q_3)\delta^4(q_2+q_3-q_4)\delta^4(q_4+p_2-p_4)}{(q_1^2-m_1c^2)(q_2^2-m_2c^2)(q_3^2-m_3c^2)(q_4^2-m_4c^2)} d^4q_1 d^4q_2 d^4q_3 d^4q_4$$

So, all our factors of 2π cancelled nicely...
now, note:

NB: there are no singularities, since virtual particles do not lie on their mass shells.

$$m_1 = m_4 = m_C \quad \& \quad m_2 = m_A \quad \& \quad m_3 = m_B$$

To integrate over q_1, because the first delta function requires $q_1 = p_1 - p_3$, so, we now have

$$-i\mathcal{M} = \frac{g^4}{((p_1-p_3)^2 - m_Cc^2)} \int \frac{\delta^4(p_1-p_3-q_2-q_3)\,\delta^4(q_2+q_3-q_4)\,\delta^4(q_4+p_2-p_4)\, d^4q_2\, d^4q_3\, d^4q_4}{(q_2^2-m_Ac^2)(q_3^2-m_Bc^2)(q^4-m_Cc^2)}$$

Integrating over q_4, we see that $q_4 \Rightarrow p_4 - p_2$ because of the delta function:

$$-i\mathcal{M} = \frac{g^4}{((p_1-p_3)^2-m_Cc^2)((p_4-p_2)^2-m_Cc^2)} \int \frac{\delta^4(p_1-p_3-q_2-q_3)\,\delta^4(q_2+q_3-p_4+p_2)}{(q_2^2-m_Ac^2)(q_3^2-m_Bc^2)} d^4q_2\, d^4q_3$$

For q_2, the first delta function tells us that $q_2 \rightarrow p_1 - p_3 - q_3$, so, once more:

the q_3's cancelled!

$$-i\mathcal{M} = \frac{g^4}{((p_1-p_3)^2-m_Cc^2)((p_4-p_2)^2-m_Cc^2)} \int \frac{\delta^4(p_1-p_3+p_2-p_4)\, d^4q_3}{((p_1-p_3-q_3)^2-m_Ac^2)(q_3^2-m_Bc^2)}$$

so, as promised, we have a delta function which enforces total conservation of energy and momentum.

6) Erase this delta function:

$$\mathcal{M} = i\left(\frac{g}{2\pi}\right)^4 \frac{1}{((p_1-p_3)^2-m_Cc^2)((p_4-p_2)^2-m_Cc^2)} \int \frac{d^4q}{((p_1-p_3-q)^2-m_Ac^2)(q^2-m_Bc^2)}$$

(I dropped the subscript...)

Problems arise with this integral, however, since it is logarithmically divergent for large q.

But, since particle theory couldn't just idle, we need a way to "sweep the infinities under the rug..."

We must do a renormalization:

This involves introducing a factor of $\dfrac{-M^2c^2}{(q^2-M^2c^2)}$ — M is the cutoff mass (very large) & will be taken to ∞.

The integral can be calculated and separated into 2 parts:

NB: goes to 1 as $M\to\infty$.

$$\mathcal{M} = \mathcal{M}_{finite} + \mathcal{M}_{infinite}$$

see below

$\mathcal{M}_{finite}$: $\not\exists\, f(M)$

$\mathcal{M}_{infinite}$: $\propto \ln M$ (divergent at $M\to\infty$)

Here, all the M-dependent terms appear as additions to the mass & to g:

$$m_{physical} = m + \delta m \qquad g_{physical} = g + \delta g$$

m: finite; δm: infinite; g: finite; δg: infinite

these are weird, but apparently m & g (which we saw in the Feynman rules) here

Comparable infinities... see Griffiths's paper.

Also, the $\mathcal{M}_{finite}$ terms contribute to alterations in mass & coupling const., but usually slight. Nonetheless, in QED → Lamb shift
in QCD → asymptotic freedom

} these are seen because of the mass dependence on the 4-momenta...

T-Hooft: all gauge theories are renormalizable

→ chromodynamics
electroweak
quantum electrodynamics

→ Which means that if all the infinities arising from higher-order can be treated as above...