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Presentation Notes
First Half of 6.2

I Fermi's Golden Rule
II Golden Rule for Decays
III Decay Into Two Massless Secondaries
IV General Two-Body Decay

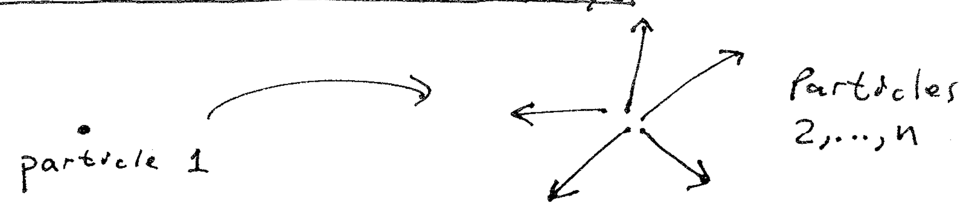
I Fermi's Golden Rule

$$\text{Transition Rate} = \frac{2\pi}{\hbar} |\mathcal{M}|^2 \times (\text{phase space})$$

From Feynman Diagrams

From Kinematic Considerations. Enforces energy and momentum conservation

II. Golden Rule for Decays



$$1 \longrightarrow 2 + 3 + \ldots + n$$

Decay Rate Formula

$$d\Gamma = |\mathcal{M}|^2 \frac{S}{2\hbar m_1} \left[\left(\frac{c\, d^3\vec{p}_2}{(2\pi)^3 2E_2} \right) \cdots \left(\frac{c\, d^3\vec{p}_n}{(2\pi)^3 2E_n} \right) \right]$$
$$\times (2\pi)^4 \delta^4 (p_1 - p_2 - \ldots - p_n) \longleftarrow \delta^4(p_i - p_f)$$

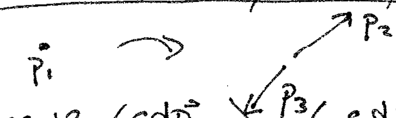
differential decay rate

$p_1 = (m_1 c, \vec{0})$ Center of Momentum Frame.

$S = \prod \frac{1}{j!}$ for each group of j identical particles

$E_i^2 = m_i^2 c^4 + \vec{p}_i^{\,2} c^2$

for a two-body decay



$$d\Gamma = \frac{S}{2\hbar m_1} |\mathcal{M}|^2 \left(\frac{c\,d\vec{p}_2}{(2\pi)^3 2E_2}\right)\left(\frac{c\,d\vec{p}_3}{(2\pi)^3 2E_3}\right) \cdot (2\pi)^4 \delta(p_1 - p_2 - p_3)$$

$$= \frac{S c^2}{32\pi^2 \hbar m_1} \frac{|\mathcal{M}|^2}{E_1 E_2} \delta^4(p_1 - p_2 - p_3)\, d\vec{p}_2\, d\vec{p}_3$$

$$S = \begin{cases} 1 & \text{for 2 distinguishable secondaries} \\ 1/2 & \text{for 2 indistinguishable secondaries} \end{cases}$$

To integrate, note that

$\mathcal{M}$ is a function of p_2, p_3

E_2 is a function of p_2

E_3 is a function of p_3

$$\Gamma = \frac{S c^2}{32\pi^2 \hbar m_1} \int \frac{|\mathcal{M}|^2}{E_2 E_3} \delta^4(p_1 - p_2 - p_3)\, d\vec{p}_2\, d\vec{p}_3$$

III Decay Into Two Massless Secondaries

$m_1 = m \qquad m_2 = m_3 = 0$

$\mathcal{M} = \mathcal{M}(\vec{p}_2, \vec{p}_3)$

Initial State

$p_1 = (mc, \vec{0})$

Final state

$p_3 = \left(\frac{E_3}{c}, \vec{p}_3\right)$

$p_2 = \left(\frac{E_2}{c}, \vec{p}_2\right)$

Kinematic Considerations

$$\delta^4(p_1 - p_2 - p_3) = \delta\left(mc - \frac{E_2}{c} - \frac{E_3}{c}\right)\delta^3\left(\vec{0} - \vec{p}_2 - \vec{p}_3\right)$$

$E_2 = |\vec{p}_2| c \qquad E_3 = |\vec{p}_3| c$

$$\Gamma = \frac{S c^2}{32\pi^2 \hbar m}\int \frac{|\mathcal{M}|^2}{|\vec{p}_2| c\, |\vec{p}_3| c}\,\delta\left(mc - |\vec{p}_2| - |\vec{p}_3|\right)\underbrace{\delta^3(-\vec{p}_2 - \vec{p}_3)}\, d\vec{p}_2\, d\vec{p}_3$$

$$\int f(\vec{p}_2, \vec{p}_3)\,\delta(-\vec{p}_2 - \vec{p}_3)\, d\vec{p}_3 = f(\vec{p}_2, -\vec{p}_2)$$

Enforces conservation of momentum $\vec{p}_3 = -\vec{p}_2$

$$\Gamma = \frac{S}{32\pi^2 \hbar m}\int \frac{|\mathcal{M}|^2}{|\vec{p}_2|^2}\,\delta(mc - 2|\vec{p}_2|)\, d^3\vec{p}_2$$

where $|\mathcal{M}|^2$ is a function of $|\vec{p}_2|$

Integration

In spherical coordinates

$$d^3\vec{p_2} = |p_2|^2 \sin\theta \, d|\vec{p_2}| \, d\theta \, d\phi$$

$$\Gamma = \frac{S}{32\pi^2\hbar m} \underbrace{\int_0^{\pi}\int_0^{2\pi} \sin\theta \, d\theta \, d\phi}_{4\pi} \int_0^{\infty} \frac{|\mathcal{M}|^2}{|\vec{p_2}|^2} \delta(mc - 2|\vec{p_2}|) \, |\vec{p_2}|^2 d|\vec{p_2}|$$

$$= \frac{S}{8\pi\hbar m} \int_0^{\infty} |\mathcal{M}|^2 \underbrace{\delta(mc - 2|\vec{p_2}|)}_{} \, d|\vec{p_2}|$$

This enforces conservation of Energy

$$\delta(mc - 2|\vec{p_2}|) = \frac{1}{2}\delta\left(|\vec{p_2}| - \frac{mc}{2}\right)$$

$$|\vec{p_2}| = \frac{mc}{2}$$

$$\boxed{\Gamma = \frac{S}{16\pi\hbar m} |\mathcal{M}^*|^2}$$

$|\mathcal{M}(\vec{p_2}, \vec{p_3})|^2$ evaluated with

$$\vec{p_2} = -\vec{p_3}$$

$$|\vec{p_2}| = \frac{mc}{2}$$

IV General Case of Two-Body Decay

Initial State — **Final State**

$(m_B c, \vec{p}_B)$

$p_i = (Mc, \vec{0})$

$p_A = (m_A c, \vec{p}_A)$

Kinematic Considerations

Again $\delta^4(p_i - p_A - p_B) = \delta\left(Mc - \frac{E_A}{c} - \frac{E_B}{c}\right) \delta^3(-\vec{p}_A - \vec{p}_B)$

$$E_A = \sqrt{m_A^2 c^4 + \vec{p}_A^{\,2} c^2} \qquad E_B = \sqrt{m_B^2 c^4 + \vec{p}_B^{\,2} c^2}$$

Again $\int f(\vec{p}_A, \vec{p}_B)\, \delta(-\vec{p}_A - \vec{p}_B)\, d^3\vec{p}_A\, d^3\vec{p}_B = \int f(\vec{p}_A, -\vec{p}_A)\, d^3\vec{p}_A$

$$\Gamma = \frac{S c^2}{32\pi^2 \hbar M} \int \frac{|\mathcal{M}|^2\, \delta\left(Mc - \sqrt{m_A^2 c^2 + \vec{p}_A^{\,2}} - \sqrt{m_B^2 c^2 + \vec{p}_A^{\,2}}\right)}{\sqrt{m_A^2 c^4 + \vec{p}_A^{\,2} c^2}\ \sqrt{m_B^2 c^4 + \vec{p}_A^{\,2} c^2}}\, d^3\vec{p}_A$$

Integration

Again $|\mathcal{M}(\vec{p}_A, -\vec{p}_A)|^2 = |\mathcal{M}|^2(|\vec{p}_A|)$

so we may change to spherical coordinates

$$d^3\vec{p}_A = |\vec{p}_A|^2 \sin\theta\, d|p_A|\, d\theta\, d\varphi$$

$$\Gamma = \frac{S}{32\pi^2 \hbar M} \int_0^\infty \frac{|\mathcal{M}|^2\, \delta\left(Mc - \sqrt{m_A^2 c^2 + \vec{p}_A^{\,2}} - \sqrt{m_B^2 c^2 + \vec{p}_A^{\,2}}\right) |p_A|^2}{\sqrt{m_A^2 c^2 + |\vec{p}_A|^2}\ \sqrt{m_B^2 c^2 + p_A^2}}\, d|p_A| \underbrace{\int d\Omega}_{4\pi}$$

$$= \frac{S}{8\pi \hbar M} \int_0^\infty \frac{|\mathcal{M}|^2\, \delta\left(Mc - \sqrt{m_A^2 c^2 + \vec{p}_A^{\,2}} - \sqrt{m_B^2 c^2 + \vec{p}_A^{\,2}}\right) |p_A|^2}{\sqrt{m_A^2 c^2 + \vec{p}_A^{\,2}}\ \sqrt{m_B^2 c^2 + \vec{p}_B^{\,2}}}\, d|p_A|$$

Change of Variables

define
$$E \equiv c\left[\sqrt{m_A^2c^2+\vec{p}_A^{\,2}}+\sqrt{m_B^2c^2+\vec{p}_A^{\,2}}\right]$$
Total Energy In final state

$$dE = c\left[\frac{|\vec{p}_A|}{\sqrt{m_A^2c^2+\vec{p}_A^{\,2}}}+\frac{|\vec{p}_A|}{\sqrt{m_B^2c^2+\vec{p}_A^{\,2}}}\right]d|\vec{p}_A|$$

$$= c|\vec{p}_A|\left[\frac{\sqrt{m_A^2c^2+\vec{p}_A^{\,2}}+\sqrt{m_B^2c^2+\vec{p}_A^{\,2}}}{\sqrt{m_A^2c^2+\vec{p}_A^{\,2}}\sqrt{m_B^2c^2+\vec{p}_A^{\,2}}}\right]d|\vec{p}_A|$$

$$= \frac{E|\vec{p}_A|\;d|\vec{p}_A|}{\sqrt{m_A^2c^2+\vec{p}_A^{\,2}}\sqrt{m_B^2c^2+\vec{p}_A^{\,2}}}$$

$|\vec{p}_A| = 0 \rightarrow E = m_Ac^2+m_Bc^2$

Then
$$\Gamma = \frac{S}{8\pi\hbar M}\int_{(m_A+m_B)c^2}^{\infty}|\mathcal{M}|^2\,\frac{|\vec{p}_A|}{E}\,\underbrace{\delta\left(Mc-\frac{E}{c}\right)}\,dE$$

conservation of energy Enforced by delta function

Note $\delta(Mc - E/c) = c\,\delta(E-Mc^2)$

If $\frac{E}{c} > Mc^2$ over the entire domain of integration then

$$\Gamma = 0$$

This corresponds to. $m_A+m_B > M$

otherwise

$$\Gamma = \frac{S}{8\pi\hbar M}|\mathcal{M}|^2\frac{|\vec{p}_A|c}{Mc^2}$$

where $|\vec{p}_0|$ is the momentum for which the final energy is Mc^2

$$\boxed{\Gamma = \frac{S|\vec{p}_0|}{8\pi\hbar c M^2}|\mathcal{M}|^2}$$

Value of $|\vec{p}_0|$

$$\frac{E_i}{c} = \frac{E_f}{c}$$

$$Mc = \sqrt{m_A^2c^2 + |\vec{p}_0|^2} + \sqrt{m_B^2c^2 + |\vec{p}_0|^2}$$

$$M^2c^2 = m_A^2c^2 + m_B^2c^2 + 2|\vec{p}_0|^2 + 2\sqrt{m_A^2m_B^2c^4 + m_A^2|\vec{p}_0|^2c^2 + m_B^2|\vec{p}_0|^2c^2 + |\vec{p}_0|^4}$$

$$\frac{(M^2 - m_A^2 - m_B^2)c^2 - 2|p_0|^2}{2} = \sqrt{|\vec{p}_0|^4 + (m_A^2 + m_B^2)|\vec{p}_0|^2c^2 + m_A^2m_B^2c^4}$$

$$\frac{(M^2 - m_A^2 - m_B^2)^2c^4 - 4(M^2 - m_A^2 - m_B^2)c^2|\vec{p}_0|^2 + 4|\vec{p}_0|^4}{4}$$

$$= |\vec{p}_0|^4 + (m_A^2 + m_B^2)|p_0|^2c^2 + m_A^2m_B^2c^4$$

$$\frac{(M^2 - m_A^2 - m_B^2)^2c^4}{4} - M^2|\vec{p}_0|^2c^2 = m_A^2m_B^2c^4$$

$$M^2|\vec{p}_0|^2 = \left\{(M^2 - m_A^2 - m_B^2)^2 - 4m_A^2m_B^2\right\}\frac{c^2}{4}$$

$$|\vec{p}_0|^2 = \frac{c^2}{4M^2}\left(M^4 + m_A^4 + m_B^4 + 2m_A^2m_B^2 - 2M^2m_A^2 - 2M^2m_B^2 - 4m_A^2m_B^2\right)$$

$$\boxed{|\vec{p}_0| = \frac{c}{2M}\sqrt{M^4 + m_A^4 + m_B^4 - 2M^2m_A^2 - 2M^2m_B^2 - 2m_A^2m_B^2}}$$

verification of formula for
two massless secondaries

$$|\vec{p}_0| = \frac{c}{2m}\sqrt{m^4} = \frac{mc}{2}$$

$$\Gamma = \frac{S\,|\vec{p}_0|}{8\pi\hbar c\, m^2}\,|\mathcal{M}|^2$$

$$= \frac{S}{16\pi\hbar m}\,|\mathcal{M}|^2 \quad ✓$$