The Golden Rule for Scattering. By Marshall Tabetah

The transition rate for a given process is determined by the amplitude and the phase space.

$$\text{transition rate} = \frac{2\pi}{\hbar} |M|^2 \times (\text{phase space})$$

Suppose particles 1 and 2 collide, producing particles $3, 4, \ldots, n$:

$$1 + 2 \rightarrow 3 + 4 + \cdots + n$$

The crosssection is given by

$$d\sigma = |M|^2 \frac{\hbar^2 S}{4\sqrt{(P_1 \cdot P_2)^2 - (m_1 m_2 c^2)^2}} \left[\left(\frac{c\, d^3\bar{p}_3}{(2\pi)^3 2E_3}\right)\left(\frac{c\, d^3 p_4}{(2\pi)^3 2E_4}\right) \cdots \left(\frac{c\, d^3 p_n}{(2\pi)^3 2E_n}\right) \right] * (2\pi)^4 * \delta^4(P_1 + P_2 - P_3 - P_4 - \cdots - P_n)$$

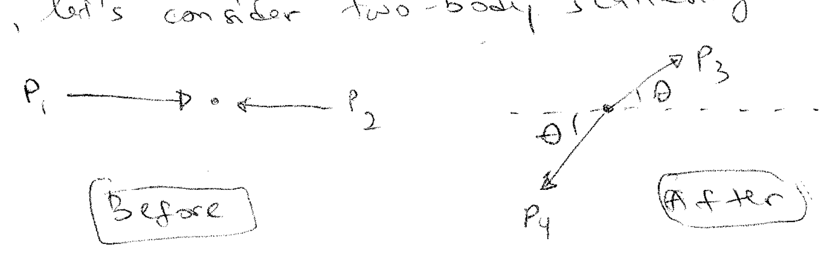
where the momentum of particle i falls in the range dp_i.

$P_i = (E_i/c, \vec{P}_i)$ – is the four-momentum of i^{th} particle.

$E_i = c\sqrt{m_i^2 c^2 + \bar{p}_i^2}$

S – statistical factor ($\frac{1}{j!}$ for each group of j identical particles in the final states.

As an example, let's consider two-body scattering in the CM frame:



$$1 + 2 \rightarrow 3 + 4$$

In the CM frame, $P_2 = -P_1 \Rightarrow P_1 \cdot P_2 = E_1 E_2/c^2 + \bar{P}_1^2$

$$\Rightarrow (P_1 \cdot P_2)^2 = (m_1 m_2 c^2)^2 + (E_1 + E_2)|P_1|/c$$

Proof : $(P_1 \cdot P_2)^2 = (m_1 m_2 c^2)^2 + (E_1+E_2)|P_1|/c$

$$(P_1 \cdot P_2) = E_1E_2/c^2 + P_1^2$$

$$\Rightarrow (P_1 \cdot P_2)^2 = (E_1E_2/c^2 + P_1^2)^2 = \frac{E_1^2E_2^2}{c^4} + P_1^4 + \frac{2E_1E_2P_1^2}{c^2}$$

$$E_1^2 = (P_1^2 + m_1^2c^2)c^2 \Rightarrow m_1^2c^2 = (E_1^2/c^2 - P_1^2)$$

$$E_2^2 = (P_2^2 + m_2^2c^2)c^2 \Rightarrow m_2^2c^2 = (E_2^2/c^2 - P_2^2)$$

$$(P_1 \cdot P_2)^2 - (m_1m_2c^2)^2 = \frac{E_1^2E_2^2}{c^4} + P_1^4 + \frac{2E_1E_2P_1^2}{c^2} - (E_1^2/c^2 - P_1^2)(E_2^2/c^2 - P_2^2)$$

$$= \cancel{\frac{E_1^2E_2^2}{c^4}} + \underbrace{(\cancel{P_1^4} - P_1^4)}_{0} + \frac{2E_1E_2P_1^2}{c^2} - \cancel{\frac{E_1^2E_2^2}{c^4}} + \frac{E_1^2P_2^2}{c^2} + \frac{P_1^2E_2^2}{c^2}$$

$$= \frac{2E_1E_2P_1^2}{c^2} + \frac{E_1^2P_1^2}{c^2} + \frac{P_1^2E_2^2}{c^2} = \frac{1}{c^2}\left[(E_1+E_2)^2P_1^2\right]$$

So, $(P_1 \cdot P_2)^2 = (m_1m_2c^2)^2 + \frac{1}{c^2}\left[(E_1+E_2)^2P_1^2\right]$

$$\Rightarrow \boxed{\sqrt{(P_1 \cdot P_2)^2 - (m_1m_2c^2)^2} = \frac{1}{c}(E_1+E_2)|P_1|}$$

Thus $$d\sigma = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|M|^2 c}{(E_1+E_2)|p_1|} \frac{d^3p_3\, d^3p_4}{E_3 E_4} \delta^4(p_1+p_2-p_3-p_4)$$

$$\delta^4(p_1+p_2-p_3-p_4) = \delta\left(\frac{E_1+E_2-E_3-E_4}{c}\right) \delta^3(-\vec{p}_3-\vec{p}_4)$$

Express $E_i = c\sqrt{m_i^2c^2+\vec{p}_i^{\,2}}$

and carry out the p_4 integral, and also $p_4 \to -p_3$

$$\Rightarrow d\sigma = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S|M|^2c^2}{(E_1+E_2)|p_1|} \times \frac{\delta\left((E_1+E_2)/c - \sqrt{m_3^2c^2+p_3^2} - \sqrt{m_4^2c^2+p_3^2}\right)}{\sqrt{m_3^2c^2+p_3^2}\,\sqrt{m_4^2c^2+p_3^2}} d^3p_3$$

$$d^3\vec{p}_3 = \vec{p}_3^{\,2}\,dp\,d\Omega \quad (\vec{p}_3^{\,2} \equiv p)$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1+E_2)|p_1|} \int_0^\infty |M|^2 \times \frac{\delta\left((E_1+E_2)/c - \sqrt{m_3^2c^2+p^2} - \sqrt{m_4^2c^2+p^2}\right)}{\sqrt{m_3^2c^2+p^2}\,\sqrt{m_4^2c^2+p^2}} p^2\,dp$$

Compare the $\frac{d\sigma}{d\Omega}$ expression to

$$\Gamma = \frac{S}{8\pi\hbar m_1} \int_0^\infty \frac{|M|^2\, \delta\left(m_1c - \sqrt{m_2^2c^2+p^2} - \sqrt{m_3^2c^2+p^2}\right)}{\sqrt{m_2^2c^2+p^2}\,\sqrt{m_3^2c^2+p^2}} p^2\,dp$$

$$\Rightarrow \Gamma = \frac{S|p|}{8\pi\hbar m_1^2 c}|M|^2$$

Let's make the replacements

$m_1c \to (E_1+E_2)/c \Rightarrow m_1 \to (E_1+E_2)/c^2$

$m_2 \to m_3$

$m_3 \to m_4$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\hbar^3}{8\pi} \frac{1}{|p_1|c}\Gamma = \frac{\hbar^3}{8\pi}\frac{1}{|p_1|c} \frac{S|p_f|}{8\pi\hbar m_1^2 c}|M|^2$$

$m_1^2 \to (E_1+E_2)^2/c^4$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\hbar^3}{\hbar\, 8\pi} \frac{|P_f|}{|P_1|} \frac{S|M|^2}{8\pi c^2 (E_1+E_2)^2/c^4} = \frac{\hbar^2 c^2}{(8\pi)^2} \frac{|P_f|}{|P_1|} \frac{S|M|^2}{(E_1+E_2)^2} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|M|^2}{(E_1+E_2)^2} \frac{|P_f|}{|P_1|}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|M|^2}{(E_1+E_2)^2} \frac{|P_f|}{|P_1|}}$$ is the differential cross section.

so that $|P_f| = |P_3|$

$[\sigma] = 1$ barn $= 10^{-24}\,\text{cm}^2$

$\left[\frac{d\sigma}{d\Omega}\right] =$ barns per steradian

M has dimensions $(mc)^{4-n}$ where n is the number of particles involved in the process.

In our case, M is dimensionless.