The Golden Rule : Golden Rule for Decays

§ 6.2 → p 194-

The transition rate for a given process is a function of the amplitude ($\mathcal{M}$) & phase space & is given by Fermi's Golden Rule (1st derived by P.A.M. Dirac)

$$\text{transition rate} = \frac{2\pi}{\hbar} |\mathcal{M}|^2 \times (\text{phase space}) \tag{1}$$

amplitude (matrix elements) density of states

Golden Rule for decays:

One particle decays into several other particles:

$1 \to 2 + 3 + 4 + \cdots + n$

The decay rate is: This is actually the <u>differential</u> decay rate

$$d\Gamma = |\mathcal{M}|^2 \frac{S}{2\hbar m_1} \left[\left(\frac{c\, d^3\vec{p}_2}{(2\pi)^3 2E_2} \right) \left(\frac{c\, d^3\vec{p}_3}{(2\pi)^3 2E_3} \right) \cdots \left(\frac{c\, d^3\vec{p}_n}{(2\pi)^3 2E_n} \right) \right] \times (2\pi)^4 \delta^4(p_1 - p_2 - p_3 \cdots p_n) \tag{2}$$

The p_i are the 4-momenta of the particles:

$$p_i = \begin{pmatrix} E_i/c \\ \vec{p}_i \end{pmatrix}$$

The decaying particle (particle 1) is presumed to be at rest

$$p_1 = \begin{pmatrix} m_1 c \\ 0 \end{pmatrix}$$

Integration over the outgoing momenta yields a total decay rate. If there are only two outgoing particles the rate is:

$$\Gamma = \frac{S}{\hbar m_1} \left(\frac{c}{4\pi} \right)^2 \frac{1}{2} \int \frac{|\mathcal{M}|^2}{E_2 E_3} \delta^4(p_1 - p_2 - p_3)\, d^3\vec{p}_2\, d^3\vec{p}_3 \tag{3}$$

* In general $|\mathcal{M}|^2$ is a function of the momenta & cannot be taken outside the integral *

Example 6.5

Take the process

$$\pi^0 \to \gamma + \gamma \quad , \text{ where } m_{\pi^0} = m$$

Find the decay rate;

$$\Gamma = \frac{S}{\hbar m}\left(\frac{c}{4\pi}\right)^2 \frac{1}{2}\int \frac{|\mathcal{M}|^2}{E_2E_3}\delta^4(p_1 - p_2 - p_3)\, d^3\bar{p}_2\, d^3\bar{p}_3$$

1^{st}, look at the δ function: $\bar{p}_1 = 0 \quad m_2, m_3 = 0$

$$\delta^4(p_1 - p_2 - p_3) = \delta\left(mc - \frac{E_2}{c} - \frac{E_3}{c}\right)\delta^3(-\bar{p}_2 - \bar{p}_3)$$

$$= \delta(mc - |\bar{p}_2| - |\bar{p}_3|)\,\delta^3(-\bar{p}_2 - \bar{p}_3)$$

$\delta^3(-\bar{p}_2 - \bar{p}_3)$ enforces $\bar{p}_3 = -\bar{p}_2$

$$\int \frac{|\mathcal{M}|^2}{E_2E_3}\delta(mc - |\bar{p}_2| - |\bar{p}_3|)\delta^3(-\bar{p}_2 - \bar{p}_3)\, d^3\bar{p}_2\, d^3p_3 \Rightarrow \text{(do } p_3 \text{ integral)}$$

$$= \frac{1}{c^2}\int \frac{|\mathcal{M}|^2}{|\bar{p}_2||\bar{p}_3|}\delta(mc - |\bar{p}_2| - |\bar{p}_3|)\delta^3(-\bar{p}_2 - \bar{p}_3)\, d^3\bar{p}_2\, d^3\bar{p}_3$$

$$= \frac{1}{c^2}\int \frac{|\mathcal{M}|^2}{|\bar{p}_2|^2}\delta(mc - 2|\bar{p}_2|)\, d^3\bar{p}_2$$

$$d^3\bar{p}_2 = |\bar{p}_2|^2\, d|\bar{p}_2|\, d\Omega$$

$$= \frac{4\pi}{c^2}\int_0^\infty |\mathcal{M}|^2\delta(mc - 2|\bar{p}_2|)\, d|\bar{p}_2|$$

$$\Gamma = \frac{S}{8\pi\hbar m}\int_0^\infty |\mathcal{M}|^2\delta(mc - 2|\bar{p}_2|)\, d|\bar{p}_2|$$

$\star\ \delta(2(mc/2 - |\bar{p}_2|)) = \frac{1}{2}\delta\left(\frac{mc}{2} - |\bar{p}_2|\right)\star$

$$\Gamma = \frac{S}{8\pi\hbar m}\int_0^\infty \frac{1}{2}|\mathcal{M}|^2\delta\left(\frac{mc}{2} - |\bar{p}_2|\right) d|\bar{p}_2|$$

$$\Gamma = \frac{S}{16\pi\hbar m}|\mathcal{M}|^2 \quad \text{w/ } \bar{p}_2 = -\bar{p}_3 \;;\; |\bar{p}_2| = \frac{mc}{2}$$

$\star\ S = \frac{1}{j!} = \frac{1}{2!} = \frac{1}{2}\ \star$

Example 6.6

Now consider

$$1 \to 2 + 3 \qquad \text{where } m_2, m_3 \neq 0$$

Find the decay rate

$$\Gamma = \frac{S}{\hbar m}\left(\frac{c}{4\pi}\right)^2 \frac{1}{2}\int \frac{|\mathcal{M}|^2}{E_2 E_3}\,\delta^4(p_1 - p_2 - p_3)\, d^3\bar{p}_2\, d^3\bar{p}_3$$

$$\bar{p}_1 = 0 \qquad E_2^2 = (m_2c^2)^2 + (\bar{p}_2 c)^2 \qquad E_3^2 = (m_3c^2)^2 + (\bar{p}_3 c)^2$$

$$\delta^4(p_1 - p_2 - p_3) = \delta\left(mc - \sqrt{m_2c^2 + \bar{p}_2^2} - \sqrt{m_3c^2 + \bar{p}_3^2}\right)\delta^3(-\bar{p}_2 - \bar{p}_3)$$

$$\Gamma = \frac{S}{\hbar m}\left(\frac{c}{4\pi}\right)^2 \frac{1}{2}\frac{1}{c^2}\int \frac{|\mathcal{M}|^2\,\delta\left(mc - \sqrt{m_2c^2 + \bar{p}_2^2} - \sqrt{m_3c^2 + \bar{p}_2^2}\right)}{\sqrt{m_2^2c^2 + \bar{p}_2^2}\sqrt{m_3^2c^2 + \bar{p}_2^2}}\, d^3\bar{p}_2$$

$$d^3\bar{p}_2 = |\bar{p}_2|^2\, d|\bar{p}_2|\, d\Omega = \rho^2\, d\rho\, d\Omega \quad \text{for } \rho = |\bar{p}_2|$$

$$\Gamma = \frac{S}{8\pi\hbar m}\int_0^\infty \frac{|\mathcal{M}|^2\,\delta\left(mc - \sqrt{m_2c^2 + \rho^2} - \sqrt{m_3c^2 + \rho^2}\right)}{\sqrt{m_2^2c^2 + \rho^2}\sqrt{m_3c^2 + \rho^2}}\,\rho^2\, d\rho$$

Define the total energy of outgoing particles

$$E \equiv c\left(\sqrt{m_2^2c^2 + \rho^2} + \sqrt{m_3^2c^2 + \rho^2}\right)$$

$$dE = \frac{E\rho}{\sqrt{m_2^2c^2 + \rho^2}\sqrt{m_3^2c^2 + \rho^2}}\, d\rho$$

$$\Gamma = \frac{S}{8\pi\hbar m}\int_{(m_2+m_3)c^2}^{\infty} |\mathcal{M}|^2 \frac{\rho}{E}\,\delta\left(mc - \frac{E}{c}\right) dE$$

$$\delta\left(mc - \frac{E}{c}\right) = c\,\delta(mc^2 - E) \qquad \text{* If } m_1 < m_2 + m_3 \;\; \Gamma = 0! \text{ *}$$

$$\Gamma = \frac{Sc}{8\pi\hbar m}|\mathcal{M}|^2\frac{\rho_0}{mc^2} \qquad \text{where } \rho_0 \text{ satisfies } E = mc^2 = c\left(\sqrt{m_2^2c^2 + \rho_0^2} + \sqrt{m_3^2c^2 + \rho_0^2}\right)$$

$$\rho_0 = |\bar{p}_2| = \frac{c}{2m}\sqrt{m^4 + m_2^4 + m_3^4 - 2m^2m_2^2 - 2m^2m_3^2 - 2m_2^2m_3^2}$$

$$\boxed{\Gamma = \frac{S|\bar{p}|}{8\pi\hbar m^2 c}|\mathcal{M}|^2}$$

$|\bar{p}|$ is outgoing momentum of either particle