Semiclassical Electromagnetic Casimir Self-Energies

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The electromagnetic Casimir energies of a spherical and a cylindrical cavity are analyzed semiclassically. The field theoretical self-stress of a spherical cavity with ideal metallic boundary conditions is reproduced to better than 1%. The subtractions in this case are unambiguous and the good agreement is interpreted as evidence that finite contributions from the exterior of the cavity are small. The semiclassical electromagnetic Casimir energy of a cylindrical cavity on the other hand vanishes to any order in the *real* reflection coefficients. The Casimir energy of a cylindrical cavity with a perfect metallic and infinitesimally thin boundary on the other hand is finite and negative[17]. Contrary to the spherical case and in agreement with Barton's perturbative analysis[31], the subtractions in the spectral density for the cylinder are not universal when only the interior modes of are taken into account[43]. The Casimir energy of a cylindrical cavity therefore depends sensitively on the physical nature of the boundary in the ultraviolet whereas the Casimir energy of a spherical one does not. The extension of the semiclassical approach to more realistic systems is sketched.

1. INTRODUCTION

Demonstrating that the collective interaction of atomic systems in some cases have macroscopic consequences, Casimir obtained the now famous attractive force between two neutral metallic plates[1] in terms of the boundary conditions they impose on the electromagnetic field. Half a century later, his prediction has been verified experimentally[2] to better than 1%.

Twenty years after Casimir's prediction for two parallel plates, Boyer calculated the zero-point energy of an ideal conducting spherical shell[3]. Contrary to intuition derived from the attraction between two parallel plates, the sphere tends to be expanded. Boyer's result has since been improved in accuracy and verified by a number of field theoretic methods[4,5,6,7,8] – even though there may be little hope of observing this effect experimentally in the near future[9].

Since field theoretic methods require explicit or implicit knowledge of cavity *frequencies*, they have predominantly been successfully employed to obtain the Casimir energies of classically *integrable* systems. Thus, in addition to a spherical cavity, the electromagnetic Casimir energies of dielectric slabs[10,11,12], metallic parallelepipeds[13,14,15,16] and long cylinders[7, 17,18,19,20] have been computed in this manner.

However, most systems are not integrable and often cannot even be approximated by integrable systems. It thus is desirable to develop reliable methods for estimating the Casimir energies of classically non-integrable and even chaotic systems. Balian, Bloch and Duplantier calculate Casimir energies based on a multiple scattering approximation to the Green's function[13,21]. This approach does not require knowledge of the quantum mechanical spectrum and the geometric expansion in principle is exact for sufficiently smooth and ideally metallic cavities. However, ultra-violet divergent contributions have to be subtracted at every order of the multiple scattering expansion. The relative importance of the finite remainder at each order in the multiple scattering expansion is hard to assess a priori and it in practice is often difficult to carry the expansion beyond the first few terms. In[22] a semiclassical method was proposed to estimate (finite) Casimir energies. It is based on Gutzwiller's trace formula[23] for the response

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function and is suitable for Casimir energies of hyperbolic and chaotic systems[22,24,25] with isolated classical periodic orbits. Although not exact in general, the semiclassical approximation associates the finite (Casimir) part of the vacuum energy with optical properties of the system. It captures aspects of Casimir energies that have been puzzling for some time[26]. Path integral methods[27,28,29,30] in principle allow one to obtain Casimir interactions between disjoint bodies to arbitrary precision. Due to unresolved renormalization problems, these methods have so far not been used to study the self stress of cavities. The purpose of this article is to estimate and analyze the Casimir stress of some cavities semiclassically. To compare with field theoretic results for spherical and cylindrical cavities, a semiclassical method that is adapted to classically integrable systems is employed. The robustness of the Casimir energy under small changes of the boundary conditions turns out to be of crucial importance for the semiclassical analysis.

The simplicity, transparency and surprising accuracy of the semiclassical approximation is demonstrated in Boyer's problem[3,4,5, 6,7,8], that is in determining the electromagnetic Casimir energy of a spherical cavity with an (ideal) metallic boundary. The semiclassical analysis of this problem is an order of magnitude simpler than any given previously. However, since no bounds are obtained, it at present is not possible to judge the accuracy of this approximation without comparing to exact field theoretic results[6]. It will become rather clear though, that the semiclassical analysis is accurate enough to infer the sign of the Casimir energy of a cavity by geometric arguments when the contribution from periodic orbits does not vanish. We shall see in sect. 4 that periodic orbits in fact do not contribute to the Casimir energy of a long cylindrical cavity. The somewhat surprising null-result that the Casimir energy of a cylinder[7,19,21,31,32,33] vanishes to first order in the reflection coefficients thus is readily explained by geometric optics. However, the semiclassical Casimir energy of a cylindrical cavity vanishes to all orders in the *real* reflection coefficients and thus also vanishes for an ideal metallic cavity. The discrepancy to the finite field theoretic Casimir energy of an idealized, infinitesimally thin cylindrical boundary between non-dispersive media with the same speed of light[17,18,19] can be traced to the presence of a logarithmic divergence observed by Barton[31] in his perturbative treatment of the non-ideal dilute case. The exact cancellation of this divergence in the field theoretic approach is due to the infinitesimal thickness of the assumed boundary – interior and exterior contributions to the Casimir energy in this case depend on just one common scale, the radius of the cylinder.

2. The Dual Picture: Casimir Energies of Integrable Systems in Terms of Periodic Rays

Integrable systems may be semiclassically quantized in terms of periodic paths on invariant tori[34] – in much the same manner as Bohr first quantized the hydrogen atom. Although in general not an exact transformation, classical periodic orbits on the invariant tori are *dual* to the mode frequencies in the semiclassical sense. Applying Poisson's summation formula, the semiclassical Casimir energy (SCE) due to a massless scalar may be written in terms of classical periodic orbits[23,26,35],

$$\begin{aligned}\mathcal{E}_c &= \frac{1}{2}\sum_{\mathbf{n}}\hbar\omega_{\mathbf{n}} - \text{UV subtractions}\\ &\sim \frac{1}{2\hbar^d}{\sum_{\mathbf{m}}}' e^{-\frac{i\pi}{2}\beta_{\mathbf{m}}}\int_{sp}\mathbf{dI}\, H(\mathbf{I})\, e^{2\pi i\,\mathbf{m}\cdot\mathbf{I}/\hbar}\ .\end{aligned} \tag{1}$$

The components of the d-dimensional vector $\mathbf{I}$ in Eq.(1) are the actions of a set of properly normalized action-angle variables that describe the integrable system. The exponent of the integrand in Eq.(1) is the classical action (in units of $\hbar$) of a periodic orbit that winds m_i times about the i-th cycle of the invariant torus. $H(\mathbf{I})$ is the associated classical energy and $\beta_{\mathbf{m}}$ is the Keller-Maslov index[36,37] of a class of periodic orbits identified by $\mathbf{m}$. The latter is a topological quantity that does not depend on the actions $\mathbf{I}$. To leading semiclassical order, the (primed) sum extends only over those sectors $\mathbf{m}$ with classical periodic paths of finite action (see below). The correspondence in Eq.(1) can only be argued semiclassically[23,35] and the integrals on the RHS therefore should be evaluated in stationary phase approximation (sp).

Contributions to the Casimir energy from high frequencies correspond to those from short periodic orbits in this dual picture. Divergences due to periodic classical paths of vanishing length (and thus vanishing total action) on the RHS of Eq.(1) correspond to ultra-violet divergences of the mode sum on the LHS of Eq.(1). If these divergences can be subtracted unambiguously[26, 31,38], the dependence of the vacuum energy on macroscopic properties of the system is semiclassically represented by contributions due to classical periodic orbits of finite action only. The primed sum on the RHS of Eq.(1) indicates this restriction[2]. The (divergent) Weyl contribution to the vacuum energy from the $\mathbf{m} = (0, \ldots, 0)$-sector in particular has to be subtracted. Together with an evaluation of the integrals in stationary phase, this defines the semiclassical Casimir energy (SCE). To physically interpret the SCE, one has to consider the implicit subtractions in the spectral density[21,26,31].

3. The Spherical Cavity

The semiclassical spectrum of a massless scalar is exact for a number of manifolds without boundary[39] and the definition of the SCE by the RHS of Eq.(1) coincides with the Casimir energy of zeta-function regularization in these cases. It also is exact for massless scalar fields satisfying periodic-, Neumann- or Dirichlet- boundary conditions on parallelepipeds[14,16,26] as well as for some tessellations of spheres[26,40,41]. In [22] the semiclassical approximation was argued to give the leading asymptotic behavior of the Casimir energy whenever the latter diverges as the ratio of two relevant lengths vanishes. All these criteria do not apply to the Casimir self-stress of a spherical cavity first considered by Boyer[3]. The latter is an integrable system, but the semiclassical spectrum is only asymptotically correct. There furthermore is no ratio of lengths in which one might hope to obtain an asymptotic expansion. One therefore cannot expect the semiclassical approximation to be exact in this case. It nevertheless turns out to be surprisingly accurate. The SCE is obtained by performing the integrals of Eq.(1) in stationary phase and has a very transparent interpretation in terms of periodic orbits *within* the cavity only. The sign of the SCE of a spherical cavity in particular will be quite trivially established and the good agreement supports the conjecture that the contribution from exterior modes mainly serves to cancel the ultra-violet divergences from the interior modes in the field theoretic approach[42,43]. The observed discrepancy of 1% to the field-theoretic results probably can be attributed to the error in the semiclassical estimate of low-lying eigenvalues of the Laplace operator – which is of similar magnitude. Since boundary conditions for the electromagnetic field never are ideal, many corrections of even greater magnitude would be required to describe a more realistic situation.

The electromagnetic Casimir energy of any closed cavity with a smooth and perfectly metallic boundary may be decomposed into the contribution from two massless scalar fields – one satisfying Dirichlet's, the other satisfying Neumann's boundary condition on the surface[21]. Because the surface is ideally thin and metallic, contributions to the spectral density from arbitrarily short closed paths that reflect off (either side of) the surface cancel each other. Semiclassically there is no (potentially divergent) local contribution to the Casimir energy from such an idealized surface in the electromagnetic case – its local surface tension in fact vanishes[21,44]. Note that this cancellation is special for the electromagnetic field and in general does not occur for a massless scalar field satisfying Neumann or Dirichlet boundary conditions on a spherical surface[45,46,47]. However, the following semiclassical argument indicates the absence of ultraviolet divergent contributions proportional to the "area" of an *infinitesimally thin* $(d-1)$-dimensional surface: barring other scales, the local contribution from a small surface element dA to the surface divergence is

[2] This is conceptually not so different from considering only the contribution of topologically non-trivial "instanton" sectors to the vacuum energy of a field theory.

proportional to $\hbar c f(R_i/R_j)\, dA/R^d$, where R is the principal (local) radius of curvature and f is a dimensionless function of ratios of the local curvatures only. The radii of curvature for closed and arbitrary short classical paths in the interior and exterior that reflect just once off the same point on the infinitesimally thin surface are of equal magnitude but of opposite sign, regardless of whether Neumann or Dirichlet boundary conditions are imposed at the surface. The divergent surface contributions from inside and outside the infinitesimally thin surface [with the same boundary condition on *both* of its sides] thus cancel precisely when d is odd. For spherical surfaces this cancellation has been explicitly shown in ref.[42]. In three dimensions, finite Casimir energies have also been explicitly obtained for scalar fields and an infinitesimally thin cylindrical cavity[18,43,48]. The previous argument indicates that in odd dimensions surface divergences cancel locally for any *infinitesimally thin* (and sufficiently smooth) boundary. It does not depend on the shape of the (smooth) boundary nor on whether Dirichlet or Neumann boundary conditions hold.

The only subtraction in the spectral density required for a finite Casimir energy in the electromagnetic case with idealized metallic boundary conditions is the Weyl contribution proportional to the volume of the sphere. The latter corresponds to ignoring the $\mathbf{m} = (0,0,0)$ contribution to the sum in Eq.(1). The remaining difficulty in calculating the SCE of an integrable system is a convenient choice of action-angle variables. For a massless scalar in three dimensions satisfying boundary conditions with spherical symmetry, an obvious set of actions is the magnitude of angular momentum, $I_2 = L$, one of the components of angular momentum $I_3 = L_z$ and an action I_1 associated with the radial degree of freedom.

Since the azimuthal angle of any classical orbit is constant, the energy $E = H(I_1, I_2)$ of a massless particle in a spherical cavity of radius R does not depend on $I_3 = L_z$. In terms of the previous choice of actions, the classical energy is implicitly given by,

$$\pi I_1 + I_2 \arccos\left(\frac{cI_2}{ER}\right) = \frac{ER}{c}\sqrt{1-\left(\frac{cI_2}{ER}\right)^2}\ . \quad (2)$$

The branches of the square root and inverse cosine in Eq.(2) are chosen so that I_1 is positive. It is convenient to introduce dimensionless variables

$$\lambda = 2ER/(\hbar c) \quad \text{and} \quad z = cI_2/(ER), \qquad (3)$$

for the total energy (in units of $\hbar c/(2R)$) and the angular momentum (in units of ER/c) of an orbit. Note that $z \in [0,1]$ and that the semiclassical regime formally corresponds to $\lambda \gg 1$, i.e. to wavelengths that are much less than the dimensions of the cavity. Using Eq.(2) and the definitions of Eq.(3) the angular frequency of the radial motion is $\omega^{-1} = (\partial E/\partial I_1)^{-1} = R\sqrt{1-(cI_2/(ER))^2}/(\pi c) = (R/\pi c)\sqrt{1-z^2}$.

With the help of Eq.(2) and the definitions of Eq.(3), the semiclassical expression in Eq.(1) for the Casimir energy of a massless scalar field satisfying Neumann or Dirichlet boundary conditions on a spherical surface becomes,

$$\begin{aligned}\mathcal{E} \;&=\; \frac{\hbar c}{4\pi R}\sum_{m,n\geq 0}{}' \Re\left[e^{-i\frac{\pi}{2}\beta(n,m)}\times\right.\\ &\times\left.\int_0^\infty d\lambda\lambda^3\int_0^1 dz z\sqrt{1-z^2}\; e^{i\lambda[n(\sqrt{1-z^2}-z\arccos(z))+m\pi z]}\right]\ .\end{aligned} \qquad (4)$$

The integral over I_3 has here been performed in stationary phase approximation. Because the Hamiltonian does not depend on I_3, only periodic orbits with $m_3 = 0$ contribute[26] in stationary phase. Since $-I_2 \leq I_3 \leq I_2$, one has that $\int dI_3 = 2I_2 = \lambda z$. The factor $2I_2$ accounts for the $2(l+1/2)$-degeneracy of a state with angular momentum $L = l+1/2 = I_2$]. By taking (4 times) the real part in Eq.(4) one can restrict the summations to non-negative integers and choose the positive branch of the square root- and inverse cosine- functions in the exponent[3]. The Keller-Maslov index $\beta(n,m)$ of a classical sector depends

[3]The primed sum now implies half the summand if one of the integers vanishes as well as the absence of the $m = n = 0$ term.

on whether Neumann or Dirichlet boundary conditions are satisfied on the spherical shell and will be determined presently.

For positive integers m and n, the phase of the integrand in Eq.(4) is stationary at $z = \bar{z}(n,m) \in [0,1]$ where,

$$\begin{aligned} 0 &= -n\arccos(\bar{z}) + m\pi \\ &\Rightarrow \bar{z}(n,m) = \cos(m\pi/n),\ n \geq 2m > 1\ . \end{aligned} \quad (5)$$

Restrictions on the values of m and n arise because $\arccos(\bar{z}) \in [0, \pi/2]$ on the chosen branch. The phase is stationary at classically allowed points only for sectors with $n \geq 2m > 1$. Semiclassical contributions to the integrals of other sectors arise due to the endpoints of the z-integration at $z = 0$ and $z = 1$ only. These "diffractive" contributions are of sub-leading order in an asymptotic expansion of the spectral density for large λ. Note that $m \to m+n$ amounts to the choice of another branch of the inverse cosine.

The classical action in sectors with stationary points is,

$$\begin{aligned} S_{cl}(n,m) &= \hbar\lambda n \sin(m\pi/n) \\ &= (E/c)2nR\sin(m\pi/n) = (E/c)L(n,m)\ , \end{aligned} \quad (6)$$

where $L(n,m)$ is the total length of the classical orbit. Some of these classical periodic orbits are shown in Fig. 1. The integer m in Eq.(6) gives the number of times an orbit circles the origin. The integer $n > 1$ gives the number of times an orbit touches the spherical surface. As indicated in Fig. 1, the set of classical periodic orbits in the (n,m)-sector form a caustic surface and a double covering is required for a unique phase-space description[36]. The two sheets are joined at the inner caustic [indicated by a dashed circle in Fig. 1] and at the outer spherical shell of radius R. Every orbit that passes the spherical shell n times also passes through the caustic n times. The cross-section of a bundle of rays is reduced to a point at the spherical caustic surface. The caustic thus is of second order and associated with a phase loss of π every time it is crossed. At each specular reflection off the outer shell Dirichlet boundary conditions require an additional phase loss of π whereas there is no phase change for Neumann boundary conditions. Altogether the Keller-Maslov index of sector (n,m) depends on n only and is given by,

$$\beta(n,m) = \begin{cases} 0, & \text{for Dirichlet b.c.} \\ 2n, & \text{for Neumann b.c.} \end{cases}\ . \quad (7)$$

For smooth surfaces on which the electromagnetic fields satisfies (ideal) metallic boundary conditions, the electromagnetic Casimir energy can be viewed as due to two massless scalar fields, one satisfying Dirichlet and the other Neumann boundary conditions[21]. Due to the Keller-Maslov phases of Eq.(7) only sectors (n,m) with *even* $n = 2k \geq 2m \geq 2$ contribute[21] to the SCE in leading order of the asymptotic expansion for large λ.

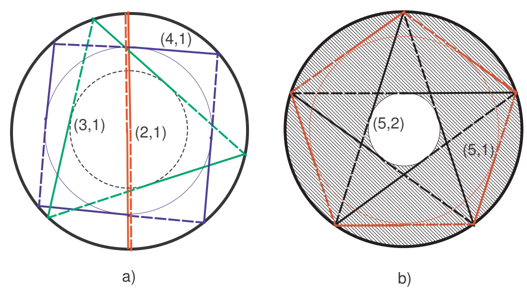

Fig.1: Classical periodic rays of a ball and solid cylinder. a) The shortest primitive rays corresponding to winding numbers $(n,m) \in \{(2,1),(3,1),(4,1)\}$. b) Primitive rays to winding numbers $(n,m) = (5,1)$ and $(5,2)$. Caustic surfaces are shown as thin circles. The dashed part of any trajectory is on one sheet and its solid part on the other of a two-sheeted covering space. The "phase space" of the $(5,2)$ sector is indicated the hatched area. Note that caustics are of 2^{nd} order for a spherical cavity but of 1^{st} order for a cylindrical one.

Note that sectors with $m = 0$ or $n = 0$ have vanishing classical action and do not contribute to the SCE. Eq.(5) implies that extremal paths in the $(n > 0, m = 0)$ sectors have maximal angular momentum $1 = \bar{z} = lc/(ER)$. These are great circles that are wholly within the spherical shell in a plane perpendicular to the one under consideration, i.e. classical orbits with $I_3 = L_z = 0$.

Because the measure of the z-integral vanishes at $z = 1$ like $\sqrt{1-z}$ these classical paths are extremal but not stationary. This can also be seen by expanding the exponent in Eq.(4) about the stationary point. For $(n > 0, m > 0)$ the curvature of the exponent at $\bar{z}(n, m)$ is finite,

$$\frac{\partial^2}{\partial \bar{z}^2}[n(\sqrt{1-\bar{z}^2} - \bar{z}\arccos(\bar{z})) + m\pi\bar{z}] = \frac{n}{\sin(\frac{m\pi}{n})}, \quad (8)$$

whereas it diverges in sectors with $m = 0$. The behavior of the exponent for $z \sim 1$ in this case is,

$$\sqrt{1-z^2} - z\arccos(z) = \tfrac{2\sqrt{2}}{3}(1-z)^{3/2} + \mathcal{O}((1-z)^{5/2}) \;. (9)$$

Quadratic fluctuations about the classical orbit with $m = 0$ thus have vanishing width and these sectors do not contribute in stationary phase approximation. To leading semiclassical accuracy, the Casimir energy of a spherical cavity with an ideal metallic boundary therefore is,

$$\begin{aligned}\mathcal{E}^{\rm ball}_{\rm EM} &\sim \frac{\hbar c}{4\pi R}\,{\rm Re}\sum_{n=1}^{\infty}(1^n + (-1)^n)\sum_{m=1}^{n/2} \\ &\times \int_0^\infty d\lambda\lambda^3 e^{in\lambda\sin(\frac{m\pi}{n})}\int_0^1 dz z\sqrt{1-z^2}\; e^{i\frac{n\lambda(z-\bar{z}(n,m))^2}{2\sin(\frac{m\pi}{n})}} \\ &\sim \frac{\hbar c}{R}\left[\sum_{k=1}^{\infty}\frac{1}{16\pi k^4} + \sum_{k=2}^{\infty}\frac{15\sqrt{2}}{256 k^4}\sum_{m=1}^{k-1}\frac{\cos(\frac{m\pi}{2k})}{\sin^2(\frac{m\pi}{2k})}\right] \\ &\sim 0.04668...\frac{\hbar c}{R}\,. \qquad (10)\end{aligned}$$

This semiclassical estimate is only about 1% larger than the best numerical value[6] $0.04617...\hbar c/R$ for the electromagnetic Casimir energy of a spherical cavity with an infinitesimally thin metallic surface. Note that the contribution from the $(2k, k)$ sectors had to be considered separately in Eq.(10) since the measure dzz vanishes at the stationary point $\bar{z}(2k, k) = \cos(\pi/2) = 0$ of the integrand, which is an endpoint of the integration domain. As can be seen in Fig. 1a), the classical rays of $(2k, k)$-sectors go back and forth between antipodes of the cavity and pass through its center – they have angular momentum $\tilde{L} = 0 = \bar{z}(2k, k)$.

The shortest primitive orbits give somewhat less than half ($1/(16\pi) \sim 0.02$) of the total SCE of the spherical cavity – much less than the 92% they contribute to the Casimir energy of parallel plates. The main reason is that contributions only drop off as $1/k^2$ rather than like $1/k^4$ as for parallel plates. The length of an orbit in the $(4, 1)$-sector (the inscribed square in Fig. 1a) furthermore is just a factor of $\sqrt{2}$ longer than an $(2, 1)$-orbit [which in turn is a factor of $1/\sqrt{2}$ shorter than an $(4, 2)$-orbit]. To estimate the magnitude of the contribution of any particular sector one has to take the available phase space as well as the ray's length into account. Thus, although the length of a $(2k, 1)$-orbit tends to $2\pi R$ for $k \to \infty$, the associated phase-space (essentially given by the volume of the shell between the boundary of the cavity and the inner caustic) decreases like $1/k^2$. This accounts for the relatively slow convergence of the sum in Eq.(10). To achieve an accuracy of 10^{-5}, the first 50 terms of the sum were evaluated explicitly and the remaining contribution was estimated using Richardson's extrapolation method.

4. The Cylindrical Cavity

The example of a spherical cavity shows that the SCE in some instances is surprisingly accurate. However, there evidently are systems without periodic classical orbits, such as the two perpendicular planes investigated in[29], or the Casimir pendulum of[49]. None of these systems is integrable, and although there are no stationary periodic classical rays, periodic rays of *extremal* (shortest) length do exist. Semiclassically, such extremal periodic rays are associated with *diffraction*[50,51]. The inclusion of diffractive contributions in the semiclassical estimate of Casimir energies has so far only been attempted for a system of spheres[52]. Below it will become evident that diffractive contributions also play a central role in the Casimir energy of a cylindrical cavity.

The Casimir energy of a dilute cylindrical gas of atoms was found to vanish in[53]. A number of calculations have confirmed that there is no contribution up to second order in the reflection coefficients for dielectrics[31,32,33] and for media where the speed of light on either side of an infinitesimally thin cylindrical boundary is the same[7,19,21]. Balian and Duplantier even conjectured that the Casimir energy of an ideal metallic cylindrical cavity may vanish[21] to all orders of the multiple reflection expansion. The non-vanishing Casimir energy of an ideal metallic cylindrical cavity[17] was subsequently reanalyzed in the framework of zeta-function regularization. It was confirmed that the Casimir energy of an ideal metallic cylinder only vanishes to leading order and that higher orders in the reflection coefficients all give a non-vanishing contribution[18]. However, some mathematical prowess is required to analytically prove the lowest order cancellation in the field-theoretic approach[33]. That a number of separate contributions should conspire to a null result without apparent physical reason has been considered by many as somewhat "mysterious"[7,12]. The suspicion that the reason could be purely geometrical is nourished by the fact that the finite part of the pair-wise Van DerWaals interaction energy of a dilute gas of atoms vanishes for a cylinder[31,53] but not for other geometries. However, a careful perturbative analysis reveals that the interaction energy of any real dilute cylindrical gas of atoms includes a logarithmic divergence in addition to divergent contributions proportional to the volume and surface area of the cylinder[31]. The subtraction of this logarithmic divergence generally is ambiguous and the Casimir energy of a cylindrical cavity depends sensitively on properties of its boundary[54] in the ultraviolet. A particular boundary (say that of an infinitesimally thin cylindrical shell separating media with the same speed of light) thus may conceivably have a finite (negative) Casimir energy, whereas a very small modification of this boundary (say in its thickness) leads to a logarithmic divergence.

The calculation below supports this possibility. The semiclassical contribution to the Casimir energy due to any periodic classical ray is found to vanish for a cylinder regardless of the reflection coefficients (without absorption). The SCE of a cylindrical cavity vanishes to *all orders* in the reflection coefficients for the same reason that the SCE of a spherical cavity is positive – due to relatively obvious optical phases. The semiclassical point of view thus gives a straightforward and physically acceptable explanation for the otherwise mysterious cancellations. It also indicates that any additional phase change at the boundary will destroy this delicate mechanism. The finite electromagnetic Casimir energy[17] of a cylinder with idealized metallic boundary conditions on the other hand is more difficult to explain semiclassically. However, contrary to a spherical cavity and in agreement with the perturbative result of[31], the semiclassical analysis of the Casimir energy of a cylindrical cavity also encounters logarithmic divergent contributions. The latter are "diffractive" end-point contributions that are ignored by the stationary phase approximation. There is reason to believe[43,48] that the subtraction of the logarithmic divergence by the contribution from "exterior" modes is the reason for the finite Casimir energy of an idealized metallic cylinder[17].

Let us now turn to the calculation of the electromagnetic SCE of a long cylindrical cavity, or rather a very thin torus with one perimeter L that is much larger than the other, $L \gg 2\pi R$. The latter is an integrable system. In the limit $R/L \to 0$, the only classical trajectories of relevance are again those of Fig. 1 and the SCE of a long cylindrical cavity can be obtained along similar lines as that of a spherical one – with some important modifications. Due to the toroidal symmetry of the (long) cylinder, the third action $I_3 = Lp_L/(2\pi)$ in this case is proportional to the conserved momentum p_L along the axis of the (thin) cylinder and in Eq.(2) the energy E must be replaced by $\sqrt{E^2 - (2\pi c I_3/L)^2}$. The second action furthermore is the angular momentum rather than just its magnitude. It again is convenient to consider dimensionless quantities for the fraction $-1 \leq x \leq 1$ of the total momentum along the axis of the cylinder, for the ratio $-1 \leq z \leq 1$ of the angular momentum to the maximal possible angular momentum of a photon within the cavity and for its energy $0 \leq \lambda < \infty$ in units of $\hbar c/(2R)$,

$$\lambda = 2ER/(\hbar c),\ z = \frac{cI_2}{ER\sqrt{1-x^2}},\ x = \frac{2\pi c I_3}{EL}\ . \quad (11)$$

Proceeding as in the spherical case, the semiclassical expression in Eq.(1) for the SCE of a massless scalar field satisfying Neumann or Dirichlet boundary conditions on a cylindrical surface becomes,

$$\mathcal{E}_{\rm cyl} = \frac{\hbar cL}{16\pi^2 R^2} \sum_{m,n\geq 0}{}' \Re\Big[e^{-i\frac{\pi}{2}\beta(n,m)} \int_0^\infty d\lambda \int_{-1}^1 dz \int_{-1}^1 dx \ \times \times \lambda^3 \sqrt{1-z^2} e^{i\lambda\sqrt{1-x^2}[n(\sqrt{1-z^2}-z\arccos(z))+m\pi z]}\Big] . \tag{12}$$

The contribution from periodic orbits that wind around the perimeter of the torus is negligible in the $R/L \to 0$ limit and has been omitted in Eq.(12). The phase of the integrand in Eq.(12) is stationary at $\bar{x} = 0$ (corresponding to $p_L = 0$) and $\bar{z}(n,m)$ given in Eq.(5). Since the domain of integration for the z-variable differs from the spherical case, sectors with $1 < m < n-1$ have non-trivial stationary points. The classical action of an (n,m)-sector is the same as for the spherical cavity and is given by Eq.(6). The fluctuations about such a classical ray on the other hand are quite different for cylindrical and spherical cavities. To quadratic order in the fluctuations about the stationary point $\bar{x} = 0, \bar{z}(n,m)$, the action for the cylinder is

$$S(n,m) \sim n\Big[\sin\frac{m\pi}{n}(1-\frac{x^2}{2}) + \frac{(z-\bar{z}(n,m))^2}{2\sin\frac{m\pi}{n}}\Big]. \tag{13}$$

The unconstrained Gaussian integrals over $z - \bar{z}(n,m)$ and x result in a factor of $2\pi/(n\lambda)$ in stationary phase approximation. Note that the phases of $\pm\pi/4$ associated with the two Gaussian integrals cancel in this case. Performing also the integral over λ in Eq.(12) finally gives,

$$\mathcal{E}_{\rm cyl} = \frac{\hbar cL}{4\pi R^2} \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} \Re \frac{-i\, e^{-i\frac{\pi}{2}\beta(n,m)}}{n^4 \sin^2\frac{m\pi}{n}} . \tag{14}$$

The crucial difference to the previous case of a spherical cavity is the phase factor of $-i$. It arises because the fluctuations of a cylindrical system have one fewer zero-mode than for a spherical one. [The Hamiltonian of a spherical cavity does not depend on $I_3 \propto L_z$, whereas it does depend on $I_3 \propto p_L$ for the cylindrical cavity. The 2×2 Hessian matrix $H_{ij} = \partial^2 H/\partial I_i \partial I_j$ with $3 > i,j > 1$ has one zero mode for a spherical cavity, but none in the cylindrical geometry. One can show[26] that this difference in zero modes implies an additional phase loss of $\pi/2$ for the periodic rays of a cylindrical cavity.] This additional phase loss ultimately is responsible for the vanishing of the SCE of a cylindrical cavity. To verify this we only need to compute the Keller-Maslov index $\beta(n,m)$ for Neumann and Dirichlet boundary conditions. The caustics of the cylindrical cavity are of first order rather than second: the cross-section of a bundle of rays becomes one-dimensional at the caustic – it is focussed to a line rather than a point. Taking into account the phase retardation by $\pi/2$ every time a ray passes a first order caustic, the analogous result to Eq.(7) for a cylindrical cavity is,

$$\beta(n,m) = \begin{cases} 3n, & \text{for Dirichlet b.c.} \\ n, & \text{for Neumann b.c.} \end{cases} \tag{15}$$

Contributions from paths with Neumann and Dirichlet boundary conditions and an odd number of reflections cancel each other and, as for the spherical cavity, only sectors to even $n = 2k = 2, 4, \ldots$ contribute to the electromagnetic SCE [this is quite generally so[21]]. Summing contributions to the electromagnetic Casimir energy from the two scalars in Eq.(14) then gives the null result

$$\mathcal{E}^{\rm EM}_{\rm cyl} = \frac{\hbar cL}{32\pi R^2} \sum_{k=1}^{\infty} \sum_{m=1}^{2k-1} \Re \frac{-i(-1)^k}{k^4 \sin^2\frac{m\pi}{2k}} = 0 . \tag{16}$$

In Eq.(16) *every* periodic orbit gives a vanishing contribution to the SCE of a cylindrical cavity. The cancellation evidently depends on a delicate relation between the optical phases. It is interesting that a small additional phase *loss* at each reflection off the surface results in a *negative* SCE for a cylindrical cavity, but that the Casimir energy vanishes as long as the above phase relations hold – even if the magnitude of the reflection coefficients is less than unity. The SCE in this sense is in line with previous results for[31,32,33,53] the Casimir energy of a dilute dielectric cylinder, and in fact supports the conjecture of Balian and Duplantier in[21]. The non-vanishing Casimir energy of a cylindrical cavity with ideal metallic boundary conditions on the other hand is not so easily explained by this semiclassical point of view.

Some insight is gained by noting that the contribution of *any* sector to the SCE of a cylindrical cavity in Eq.(12) – even sectors with non-trivial periodic classical paths – diverges. This is in marked contrast to the spherical case, where the contribution from sectors with non-trivial periodic classical paths (characterized by $n \geq 2m > 1$) is *finite*. The divergence is most readily made explicit by scaling $\lambda\sqrt{1-x^2} \to \lambda$ in the integral of Eq.(12). Without ultraviolet cutoff, the resulting x-integral in this case formally gives the factor,

$$\int_{-1}^{1} \frac{dx}{(1-x^2)^2} \sim \infty \ , \tag{17}$$

whose divergence is due to the behavior of the integrand as $x \to \pm 1$. It may be regulated by introducing an *ultraviolet* cutoff Ω of some sort for the energy integral [that is in the integral over λ]. As may be seen from Eq.(17), the regulated integral will always include terms that are *logarithmically* divergent as $\Omega \to \infty$. The subtraction of a logarithmic divergence depends on details of the cutoff and thus is sensitive to ultraviolet properties of the boundary[38]. The evaluation of (divergent) integrals in stationary phase can be considered*one* way of subtracting the divergence. Because the divergence is logarithmic, the subtraction is by no means unique in this case. The presence of such a logarithmic divergence for cylindrical cavities was first emphasized by Barton[31] in a perturbative treatment of a dilute gas of atoms, although it also is evident in the contribution from interior modes to the Casimir energy of an ideally metallic cylinder[43].

The foregoing is compatible with previous results[17,19,7] that the Casimir energy of a cylindrical cavity is finite if the speed of light inside and outside its*infinitesimally thin* boundary surface are the same. It for instance is negative for idealized metallic boundary conditions[17]. The Casimir energy in this case apparently does not suffer from any logarithmic divergences (or equivalently, from any pole ambiguities in zeta function regularization). The Casimir energy is finite for the infinitesimally thin boundary, because the logarithmic divergent contribution from interior modes is precisely cancelled by the similarly logarithmic divergent contribution from exterior modes. Since the boundary is infinitesimally thin and the speed of light is the the same, a precise cancellation is possible. The divergence reappears for a dielectric cavity in vacuum with a lower speed of light in the dielectric[54]. This occurs for a spherical as well as for a cylindrical cavity, but with an important difference: the divergence in the spherical case is not logarithmic and may be unambiguously subtracted[9]. The subtraction of the logarithmic divergence in the Casimir energy of the cylindrical cavity on the other hand requires some energy scale that describes properties of the boundary in the ultraviolet. An analogous problem would be encountered for an ideal metallic boundary of finite thickness[43] and in fact for almost any small deviations from an idealized and infinitesimally thin cylindrical boundary between two media with identical speed of light. Paradoxically, *defining* the Casimir energy of a cylindrical cavity in a manner that does not depend on the detailed ultraviolet properties of its boundary appears all but impossible.

It perhaps is worth mentioning in this regard that the Casimir energy of a massless scalar excitation *on* the two-dimensional spherical or toroidal boundaries is well-defined. For a spherical shell and a very thin torus, this Casimir energy has the same dependence on the dimensions as the Casimir energies of the corresponding cavities. For a two-sphere (S_2) and a very thin torus T_2 with $L \gg 2\pi R$ these Casimir energies are

$$\begin{aligned} \mathcal{E}_{S_2} &= 0 \qquad\qquad (18) \\ \mathcal{E}_{T_2} &= -\frac{\hbar c L}{4\pi^3 R^2}\zeta(3) \sim -0.0097\ldots\frac{\hbar c L}{R^2} \ . \end{aligned}$$

Note that these Casimir energies of a massless scalar on two-dimensional spherical and toroidal surfaces are exactly reproduced semiclassically[26,39,16]. The presence of scalar surface modes therefore does not change the Casimir energy of a spherical cavity but could very well contribute to that of a cylindrical one. The Casimir energy of a massless degree of freedom on a torus not only is of the same form, but also of the same sign and order of magnitude as the Casimir energy of an ideal metallic cylindrical cavity[17,18]. Such a contribution from massless surface modes thus might be important for a cylindrical cavity and would furthermore be difficult to separate from the contribution due to cavity modes.

5. Discussion

The semiclassical approximation to the Casimir energy of a cavity to leading order includes only contributions from quadratic fluctuations about stationary periodic classical rays. Since all periodic rays lie in the interior, the SCE of a concave cavity to leading order depends on the exterior only indirectly through reflection coefficients. Periodic classical rays furthermore are of finite length. Their contribution to the Casimir energy thus is ultraviolet finite. However, this approximation is sensible only if UV-divergent contributions to the vacuum energy can be subtracted unambiguously from the spectral density. Logarithmically divergent contributions to the vacuum energy require a subtraction scale[38]. The latter is a clear indication that the subtraction cannot be universal since it depends sensitively on the UV-properties of the boundary. Small changes in the boundary conditions in this case do not necessarily correspond to small changes in the Casimir energy. The *local* properties of a boundary the vacuum energy can be sensitive to apparently include its thickness: whereas the Casimir energy of a cylindrical cavity with an ideal and infinitesimally thin metallic boundary is finite[17] to any order in the (real) reflection coefficients[18], a logarithmic dependence on the cutoff appears in more realistic situations[31,43]. An ambiguous subtraction is also required in the semiclassical approximation. The absence of any logarithmic divergence for the infinitesimally thin boundary apparently is due to a cancellation by exterior modes. Such a cancellation of logarithmic singularities can occur when exterior and interior modes depend on precisely the same scale, the radius R of the cylindrical cavity in this case. Although the two logarithmic divergences (each proportional to $\hbar cL/R^2$ for dimensional reasons) cancel in the idealized situation, they would not if the boundary is of finite thickness.

We considered only the semiclassical Casimir energy (SCE) of a spherical and of a toroidal cavity with ideal metallic boundary conditions, that is with real reflection coefficients of unit magnitude. These are integrable systems and the SCE was derived from the "dual" description of the spectral density in terms of periodic paths on invariant tori[23,35]. The winding numbers of a periodic orbit are dual to the quantum numbers of a mode. In stationary phase approximation the SCE of a spherical cavity is *positive* and coincides with the field theoretic value for an infinitesimally thin metallic boundary to about 1%. The calculation is rather short and straightforward and leads to the convergent sum of Eq.(10). Each term in this sum may be interpreted as the contribution from a class of periodic rays. A few the shorter primitive periodic rays are depicted in Fig. 1. The contribution from any sector with classical periodic rays is finite in this case. Divergent contributions are restricted to sectors with no classical rays.

The contribution from periodic orbits to the SCE of a cylindrical cavity with an ideal metallic boundary on the other hand vanishes to all orders in the number of reflections. This occurs due to an overall phase change by an odd multiple of $\pi/2$ for any classical periodic ray. Restricting to just two reflections, this null result agrees with field theoretic calculations for infinitesimally thin metallic boundaries[7,19,?, 21,55]. The vanishing SCE appears to support the conjecture of Balian and Duplantier that the Casimir energy of a metallic cylindrical cavity may vanish. However, contrary to the spherical case, the contributions of *any* classical sector to the SCE of a cylindrical cavity diverges. Without subtraction of the UV-divergent part, the (finite) semi-classical contribution to the vacuum energy we obtained is not very meaningful. Unfortunately the divergence of the integral in Eq.(17) includes a logarithmic dependence on the cutoff. The subtraction of UV-divergent contributions to the Casimir energy of a cylinder thus is sensitive to a scale and cannot be achieved in a universal fashion. The logarithmic dependence on the cutoff was first observed by Barton[31] in his perturbative calculation of the vacuum energy for a (dilute) gas of cylindrical shape to lowest order in the fine structure constant. Semiclassically this would also correspond to considering the contribution from rays with only two reflections ($n = 2$). That the UV-subtractions are fragile and depend crucially on the UV-properties of the boundary is also observed when the speed of light within and outside an infinitesimally thin cylindrical boundary differ[54]. In the electromagnetic case, the logarithmic divergences of exterior and interior contributions to the vacuum energy of a metallic cylinder cancel for an infinitesimally thin metallic boundary[43]. However, they in general cannot be unambiguously subtracted[48].

These examples of a spherical- and cylindrical cavity show that the SCE is quite reasonable and is rather simple to calculate when the Casimir energy is robust, that is, when the subtractions do not depend on fine-tuning of the ultraviolet behavior of the boundary. The classical periodic paths that contribute to the SCE of a concave cavity in stationary phase approximation lie entirely within the cavity. Their contribution depends on the exterior of the cavity through reflection coefficients only. It has been argued for some time that a Casimir energy obtained without explicit inclusion of exterior modes (as for a parallelepiped[14,16]) is all but meaningless[12]. The criterion favored here[31,26] considers any definition of a Casimir energy reasonable (and in principle physically realizable) in which the UV-divergences of the vacuum energy have been subtracted in a universal fashion, that is without explicit reference to UV-properties of the boundary. The subtraction may (and in general will) include divergent contributions from exterior modes. The Casimir energy of a parallelepiped can be considered a case in point: as Power[56] did for just two slabs, one can always assemble (at most 8) parallelepipeds to a cube of fixed dimensions – the Casimir energy of an individual parallelepiped[14,16] in this case reflects changes in the vacuum energy of the whole cube as the four dividing planes are moved adiabatically. By moving interior surfaces of the cube (that in principle could have finite thickness), one measures only that finite part of its vacuum energy that depends on the dimensions of the individual parallelepipeds. By contrast, it is difficult to imagine that global changes in a vacuum energy are measurable (or even physically relevant) if their finiteness depends crucially on local characteristics of the system[26]. Perhaps somewhat surprisingly, the electromagnetic Casimir energy of a very long cylindrical cavity does not appear to be robust in this sense, whereas the electromagnetic Casimir energy of a spherical cavity is.

Apart from relating Casimir energies to optical properties, one of the advantages of a semiclassical description would be the possibility to model more realistic (but robust) physical systems. The previous considerations are readily extended to dielectrics by using appropriate complex and in general frequency-dependent reflection coefficients. In the case of dielectric slabs Milton has shown[12] that Lifshitz's theory[10,11] may be reproduced in this manner. Finite temperature is incorporated[57] by allowing periodic rays to also wrap around a fictitious periodic extra dimension of circumference $\hbar c/(kT)$. Finite temperature corrections thus are small if some classical periodic paths are much shorter than this circumference. At room temperature the length of a periodic ray increases by of about 7.6 microns every time it winds about the temperature direction. Temperature corrections therefore are tiny for most nanometer scale experiments[4] but could be of greater interest in some astrophysical considerations ($3^o K \equiv 1$mm). Corrections due to surface roughness generally will be more important in technological applications. Many classical models for diffusive reflection from rough surfaces exist and Lambert's Law is easily incorporated in the semiclassical approach by appropriate reflection coefficients. The dependence on the wavelength perhaps can be modelled by a stochastic term of the action that accounts for fluctuations in the length of a classical periodic orbit upon reflection from rough surfaces. Apart from an average change in length, this leads to a damping term of the form $-(\Delta L\, E/\hbar c)^2/2$ in the classical action, where $(\Delta L)^2$ is the variance in the length of the periodic orbit. Assuming that this variance is itself proportional to the length of the orbit,

[4] The claim that this correction has been measured to sufficient accuracy[2] to distinguish between different approaches has recently been disputed[58].

surface roughness can semiclassically perhaps be modelled by the modified dispersion

$$cp(E) = E + i\varepsilon E^2/(\hbar c)\ , \tag{19}$$

where ε is a typical length scale for the (stochastic) roughness of the surface. The predominant effect of the modified dispersion of Eq.(19) is that contributions to the Casimir energy from wave lengths $\lambda \ll \varepsilon$ are very much suppressed. A similar conclusion may be drawn from a recent and considerably more sophisticated analysis[59].

Acknowledgements: I am indebted to Larry Spruch for advice and numerous discussions and would like to thank K. A. Milton for a critical reading of the manuscript. His remarks led to a lengthy discussion of the very interesting results of[42] in section 3. This work is partially supported by the National Science Foundation under Grant No. PHY0555580.

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