

# Multiple-Scattering Methods in Casimir Calculations

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- D. A. R. Dalvit, F. C. Lombardo, F. D. Mazzitelli, and R. Onofrio, PRA 74, 020101(R) (2006); New J. Phys. , 240 (2006)
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- Kenneth and Klich, arXiv:0707.4017 [quant-ph] (drafted earlier?)

# Green's Function Approach

The multiple scattering approach starts from the well-known formula for the vacuum energy or Casimir energy (for simplicity here we restrict attention to a massless scalar field) ( $\tau$  is the “infinite” time that the configuration exists)  
[Schwinger, 1975]

$$E = \frac{i}{2\tau} \text{Tr} \ln G,$$

where  $G$  is the Green's function,

$$(-\partial^2 + V)G = 1, \quad +\text{BC}.$$

# Ambiguity in formula

The above formula for the Casimir energy is defined up to an infinite constant, which can be at least partially compensated by inserting a factor as do Kenneth and Klich:

$$E = \frac{i}{2\tau} \text{Tr} \ln GG_0^{-1}.$$

Here  $G_0$  satisfies, with the same boundary conditions as  $G$ , the free equation

$$-\partial^2 G_0 = 1.$$

# *T*-matrix

Now we define the *T*-matrix,

$$T = S - 1 = V(1 + G_0V)^{-1}.$$

If the potential has two disjoint parts,

$$V = V_1 + V_2$$

it is easy to show that

$$T = (V_1 + V_2)(1 - G_0T_1)(1 - G_0T_1G_0T_2)^{-1}(1 - G_0T_2),$$

where

$$T_i = V_i(1 + G_0V_i)^{-1}, \quad i = 1, 2.$$

# Interaction in terms of $T_i$ or $G_i$

Thus, we can write the general expression for the interaction between the two bodies (potentials) in two alternative forms:

$$\begin{aligned} E_{12} &= -\frac{i}{2\tau} \text{Tr} \ln(1 - G_0 T_1 G_0 T_2) \\ &= -\frac{i}{2\tau} \text{Tr} \ln(1 - V_1 G_1 V_2 G_2), \end{aligned}$$

where

$$G_i = (1 + G_0 V_i)^{-1} G_0, \quad i = 1, 2.$$

# Multipole expansion

To proceed to apply this method to general bodies, we use an even older technique, the multipole expansion. Let's illustrate this with a  $2 + 1$  dimensional version, which allows us to describe cylinders with parallel axes. We seek an expansion of the free Green's function

$$\begin{aligned} G_0(\mathbf{R} + \mathbf{r}' - \mathbf{r}) &= \frac{e^{i|\omega||\mathbf{r} - \mathbf{R} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{R} - \mathbf{r}'|} \\ &= \int \frac{dk_z}{2\pi} e^{ik_z(z - Z - z')} g_0(\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp), \end{aligned}$$

# Reduced Green's function

$$g_0(\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp) = \int \frac{(d^2 k_\perp)}{(2\pi)^2} \frac{e^{-i\mathbf{k}_\perp \cdot \mathbf{R}_\perp} e^{i\mathbf{k}_\perp \cdot (\mathbf{r}_\perp - \mathbf{r}'_\perp)}}{k_\perp^2 + k_z^2 + \zeta^2}.$$

As long as the two potentials do not overlap, so that we have  $\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp \neq 0$ , we can write an expansion in terms of modified Bessel functions:

$$g_0(\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp) = \sum_{m,m'} I_m(\kappa r) e^{im\phi} I'_m(\kappa r') e^{-im'\phi'} \\ \times \tilde{g}_{m,m'}^0(\kappa R), \quad \kappa^2 = k_z^2 + \zeta^2.$$

# Expression for $g_{m,m'}^0$

By Fourier transforming, and using the definition of the Bessel function

$$i^m J_m(kr) = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-im\phi} e^{ikr \cos \phi},$$

we easily find

$$\begin{aligned} \tilde{g}_{m,m'}^0(\kappa R) &= \frac{1}{2\pi} \int \frac{dk k}{k^2 + \kappa^2} J_{m-m'}(kR) \frac{J_m(kr) J_m(kr')}{I_m(\kappa r) I_m(\kappa r')} \\ &= \frac{(-1)^{m'}}{2\pi} K_{m-m'}(\kappa R). \end{aligned}$$

# Discrete matrix realization

Thus we can derive an expression for the interaction between two bodies, in terms of discrete matrices,

$$\mathfrak{E} \equiv \frac{E_{\text{int}}}{L} = \frac{1}{8\pi^2} \int d\zeta dk_z \ln \det \left( 1 - \tilde{g}^0 \tilde{T}_1 \tilde{g}^{0\top} \tilde{T}_2 \right),$$

where the  $\tilde{T}$  matrix elements are given by

$$\begin{aligned} \tilde{T}_{mm'} = & \int dr r d\phi \int dr' r' d\phi' I_m(\kappa r) e^{-im\phi} I_{m'}(\kappa r') e^{im'\phi'} \\ & \times T(r, \phi; r', \phi'). \end{aligned}$$

# Interaction between cylinders

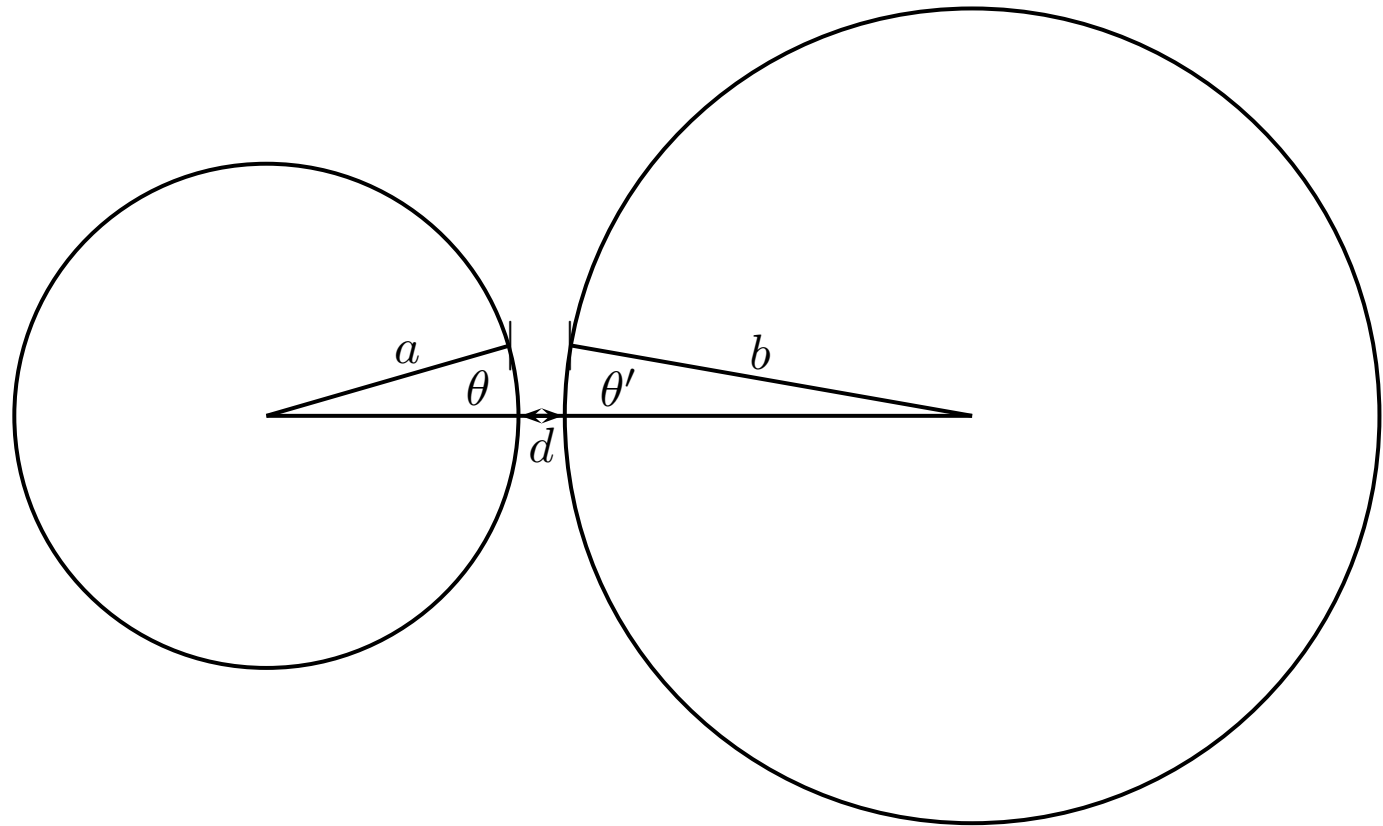


Figure 1: Geometry of two cylinders (or two spheres) with radii  $a$  and  $b$ , respectively, and distances between their centers of  $R > a + b$ .

# Semitransparent cylinders

Consider two parallel semitransparent cylinders, of radii  $a$  and  $b$ , respectively, lying outside each other, described by the potentials

$$V_1 = \lambda_1 \delta(r - a), \quad V_2 = \lambda_2 \delta(r' - b),$$

with the separation between the centers  $R$  satisfying  $R > a + b$ . It is easy to work out the scattering matrix in this situation,

$$(t_1)_{mm'} = 2\pi \lambda_1 a \delta_{mm'} \frac{I_m^2(\kappa a)}{1 + \lambda_1 a I_m(\kappa a) K_m(\kappa a)}.$$

# Cylinder interaction

Thus the Casimir energy per unit length is

$$\mathfrak{E} = \frac{1}{4\pi} \int_0^\infty d\kappa \kappa \operatorname{tr} \ln(1 - A),$$

where  $A = B(a)B(b)$ , in terms of the matrices

$$B_{mm'}(a) = K_{m+m'}(\kappa R) \frac{\lambda_1 a I_{m'}^2(\kappa a)}{1 + \lambda_1 a I_{m'}(\kappa a) K_{m'}(\kappa a)}.$$

# Weak-coupling

In weak coupling, the formula for the interaction energy between two cylinders is

$$\mathcal{E} = -\frac{\lambda_1 \lambda_2 ab}{4\pi R^2} \sum_{m, m' = -\infty}^{\infty} \int_0^{\infty} dx x K_{m+m'}^2(x) \\ \times I_m^2(xa/R) I_{m'}^2(xb/R).$$

# Power series expansion

One merely exploits the small argument expansion for the modified Bessel functions  $I_m(xa/R)$  and  $I_{m'}(xb/R)$ :

$$I_m^2(x) = \left(\frac{x}{2}\right)^{2|m|} \sum_{n=0}^{\infty} Z_{|m|,n} \left(\frac{x}{2}\right)^{2n},$$

where the coefficients  $Z_{m,n}$  are

$$Z_{m,n} = \frac{2^{2(m+n)} \Gamma\left(m + n + \frac{1}{2}\right)}{\sqrt{\pi} n! (2m + n)! \Gamma(m + n + 1)}.$$

# Closed form result

In this case we get an amazingly simple result

$$\mathfrak{E} = -\frac{\lambda_1 a \lambda_2 b}{4\pi R^2} \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{a}{R}\right)^{2n} P_n(\mu),$$

where  $\mu = b/a$ , and where by inspection we identify the binomial coefficients

$$P_n(\mu) = \sum_{k=0}^n \binom{n}{k} \mu^{2k}.$$

# Closed form result (cont.)

Remarkably, it is possible to perform the sums, so we obtain the following closed form for the interaction between two weakly-coupled cylinders:

$$\mathfrak{E} = -\frac{\lambda_1 a \lambda_2 b}{8\pi R^2} \left[ \left( 1 - \left( \frac{a+b}{R} \right)^2 \right) \left( 1 - \left( \frac{a-b}{R} \right)^2 \right) \right]^{-1/2}.$$

We note that in the limit  $R - a - b = d \rightarrow 0$ ,  $d$  being the distance between the closest points on the two cylinders, we recover the proximity force theorem in this case

$$U(d) = -\frac{\lambda_1 \lambda_2}{32\pi} \sqrt{\frac{2ab}{R}} \frac{1}{d^{1/2}}, \quad d \ll a, b.$$

The rate of approach is given by

$$\frac{\mathcal{E}}{U} \approx 1 - \frac{1 + \mu + \mu^2}{4\mu} \frac{d}{R} \approx 1 - \frac{R^2 - aR + a^2}{4a(R - a)} \frac{d}{R}.$$

$$a = b$$

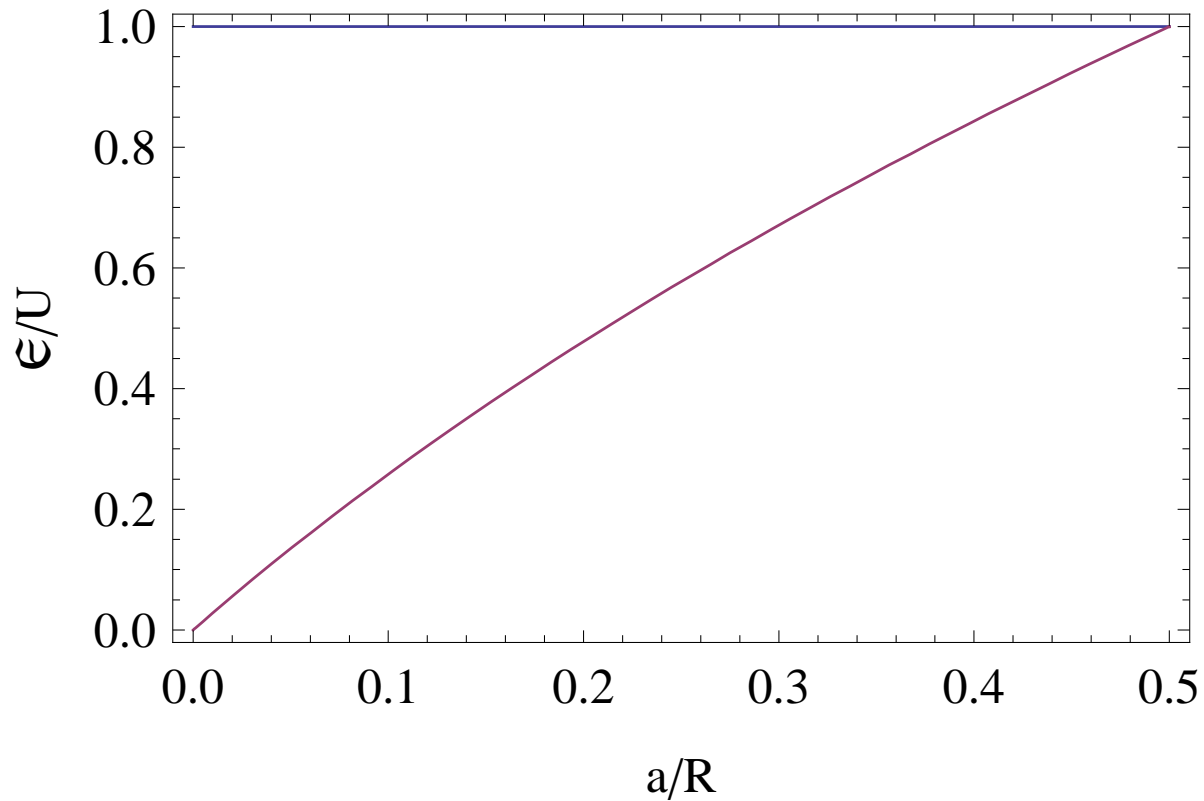


Figure 2: Plotted is the ratio of the exact interaction energy of two weakly-coupled cylinders to the proximity force approximation

$$b/a = 99$$

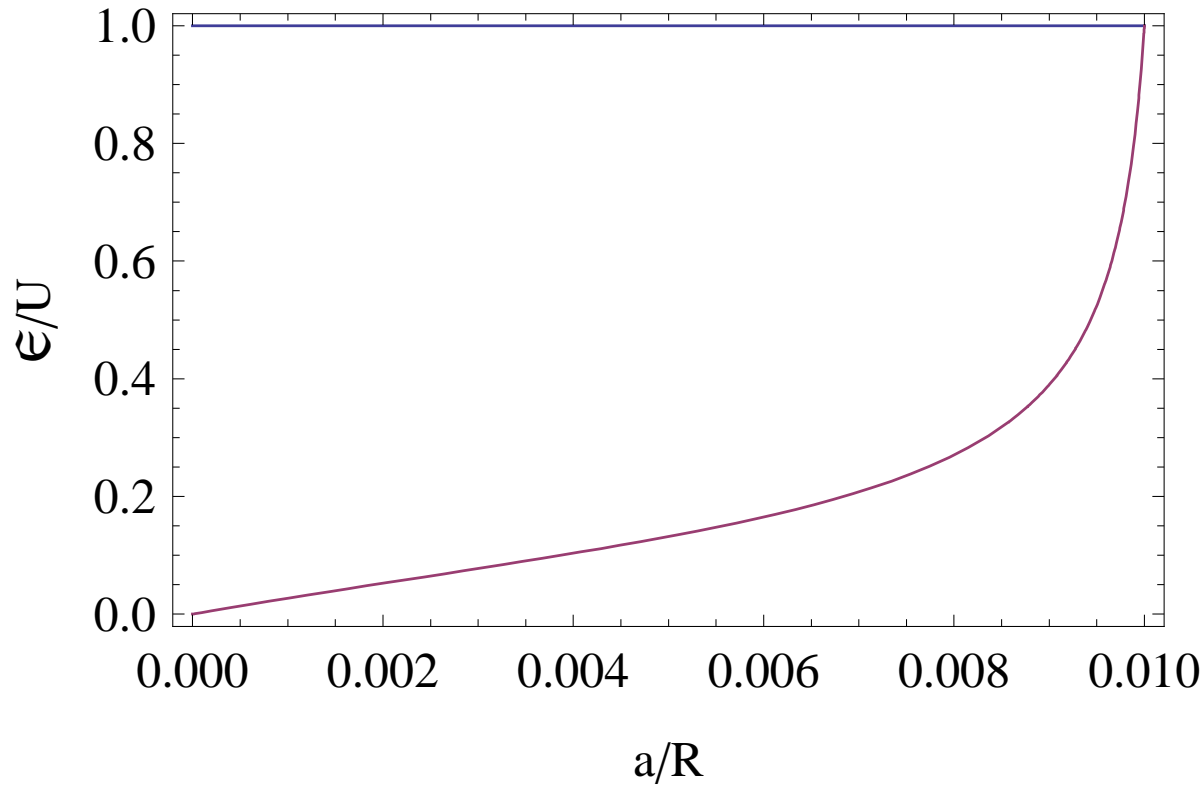


Figure 3: Plotted is the ratio of the exact interaction energy of two weakly-coupled cylinders to the proximity force approximation

# Cylinder/plane interaction

By the method of images, we can find the interaction between semitransparent cylinder and a Dirichlet plane is

$$\mathcal{E} = \frac{1}{4\pi} \int_0^\infty \kappa d\kappa \operatorname{tr} \ln(1 - B(a)),$$

where  $B(a)$  is given above. In the strong-coupling limit this result agrees with that given by Bordag, because

$$\operatorname{tr} B^s = \operatorname{tr} \tilde{B}^s, \quad \tilde{B}_{mm'} = \frac{1}{K_m(\kappa a)} K_{m+m'}(\kappa R) I_{m'}(\kappa a).$$

# Exact cylinder/plane energy

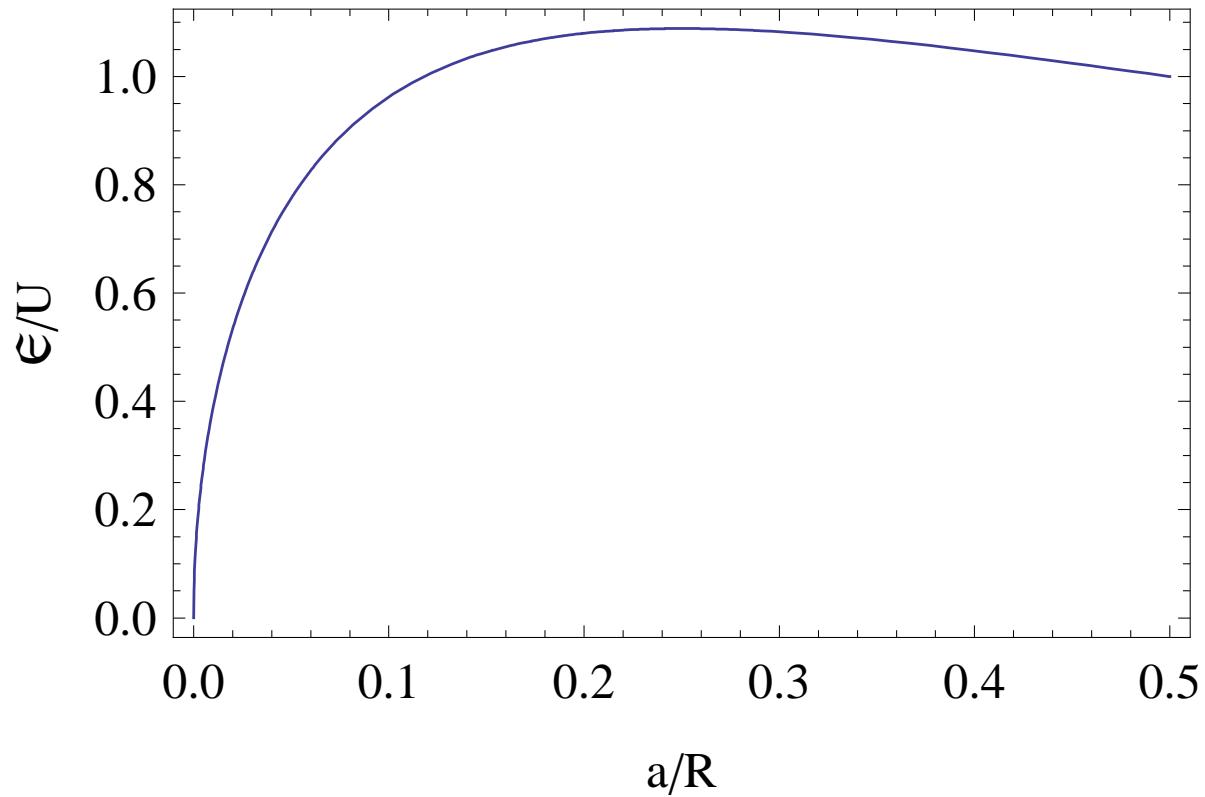
In exactly the same way, we can obtain a closed-form result for the interaction energy between a Dirichlet plane and a weakly-coupled cylinder of radius  $a$  separated by a distance  $R/2$ . The result is again quite simple:

$$\mathcal{E} = -\frac{\lambda a}{4\pi R^2} \left[ 1 - \left( \frac{2a}{R} \right)^2 \right]^{-3/2}.$$

In the limit as  $d \rightarrow 0$ , this agrees with the PFA:

$$U(d) = -\frac{\lambda}{64\pi} \frac{\sqrt{2a}}{d^{3/2}}.$$

# Comparison of PFA and exact



# 3-dimensional formalism

The three-dimensional formalism is very similar. In this case, the free Green's function has the representation

$$G_0(\mathbf{R} + \mathbf{r}' - \mathbf{r}) = \sum_{lm, l'm'} j_l(i|\zeta|r) j_{l'}(i|\zeta|r') Y_{lm}^*(\hat{\mathbf{r}}) Y_{l'm'}(\hat{\mathbf{r}}') \\ \times g_{lm, l'm'}(\mathbf{R}).$$

# Reduced Green's function

The reduced Green's function can be written in the form

$$g_{lm,l'm'}^0(\mathbf{R}) = (4\pi)^2 i^{l'-l} \int \frac{(d\mathbf{k})}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{R}}}{k^2 + \zeta^2} \frac{j_l(kr)j_{l'}(kr')}{j_l(i|\zeta|r)j_{l'}(i|\zeta|r')} \\ \times Y_{lm}(\hat{\mathbf{k}})Y_{l'm'}^*(\hat{\mathbf{k}}).$$

Now we use the plane-wave expansion once again, this time for  $e^{i\mathbf{k}\cdot\mathbf{R}}$ ,

$$e^{i\mathbf{k}\cdot\mathbf{R}} = 4\pi \sum_{l''m''} i^{l''} j_{l''}(kR) Y_{l''m''}(\hat{\mathbf{R}}) Y_{l''m''}^*(\hat{\mathbf{k}}),$$

so now we encounter something new, an integral over three spherical harmonics,

$$\int d\hat{\mathbf{k}} Y_{lm}(\hat{\mathbf{k}}) Y_{l'm'}^*(\hat{\mathbf{k}}) Y_{l''m''}^*(\hat{\mathbf{k}}) = C_{lm,l'm',l''m''},$$

# Wigner coefficients

where

$$C_{lm,l'm',l''m''} = (-1)^{m'+m''} \sqrt{\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}} \\ \times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix}.$$

The three- $j$  symbols (Wigner coefficients) here vanish unless  $l + l' + l''$  is even.

# Reduced Green's function

This fact is crucial, since because of it we can follow the previous method of writing  $j_{l''}(kR)$  in terms of Hankel functions of the first and second kind, using the reflection property of the latter,  $h_{l''}^{(2)}(kR) = (-1)^{l''} h_{l''}^{(1)}(-kR)$ , and then extending the  $k$  integral over the entire real axis to a contour integral closed in the upper half plane.

$$g_{lm,l'm'}^0(\mathbf{R}) = 4\pi i^{l'-l} \sqrt{\frac{2|\zeta|}{\pi R}} \sum_{l''m''} C_{lm,l'm',l''m''} \times K_{l''+1/2}(|\zeta|R) Y_{l''m''}(\hat{\mathbf{R}}).$$

# Casimir interaction of spheres

For the case of two semitransparent spheres that are totally outside each other,

$$V_1(r) = \lambda_1 \delta(r - a), \quad V_2(r') = \lambda_2 \delta(r' - b),$$

in terms of spherical coordinates centered on each sphere, it is again very easy to calculate the scattering matrices,

$$T_1(\mathbf{r}, \mathbf{r}') = \frac{\lambda_1}{a^2} \delta(r - a) \delta(r' - a) \times \sum_{lm} \frac{Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}')}{1 + \lambda_1 a K_{l+1/2}(|\zeta|a) I_{l+1/2}(|\zeta|a)},$$

# Scattering matrix element

and then the harmonic transform is very similar to that seen for the cylinder, ( $k = i|\zeta|$ )

$$\begin{aligned}(t_1)_{lm,l'm'} &= \int (d\mathbf{r})(d\mathbf{r}') j_l(kr) Y_{lm}^*(\hat{\mathbf{r}}) j_{l'}(kr') Y_{l'm'}(\hat{\mathbf{r}}') T_1(\mathbf{r}, \mathbf{r}') \\ &= \delta_{ll'} \delta_{mm'} (-1)^l \frac{\lambda_1 a \pi}{2|\zeta|} \frac{I_{l+1/2}^2(|\zeta|a)}{1 + \lambda_1 a K_{l+1/2}(|\zeta|a) I_{l+1/2}(|\zeta|a)}.\end{aligned}$$

# Interaction energy

Let us suppose that the two spheres lie along the  $z$ -axis, that is,  $\mathbf{R} = R\hat{\mathbf{z}}$ . Then we can simplify the expression for the energy somewhat by using  $Y_{lm}(\theta = 0) = \delta_{m0}\sqrt{(2l+1)/4\pi}$ . The formula for the energy of interaction becomes

$$E = \frac{1}{2\pi} \int_0^\infty d\zeta \operatorname{tr} \ln(1 - A),$$

where the matrix

$$A_{lm,l'm'} = \delta_{m,m'} \sum_{l''} B_{ll''m}(a) B_{l''l'm}(b)$$

$$B_{ll'm}(a) = \frac{\sqrt{\pi}}{\sqrt{2\zeta R}} i^{-l+l'} \sqrt{(2l+1)(2l'+1)} \sum_{l''} (2l''+1) \\ \times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix} \frac{K_{l''+1/2}(\zeta R) \lambda_1 a I_{l''+1/2}^2(\zeta a)}{1 + \lambda_1 a I_{l''+1/2}(\zeta a) K_{l''+1/2}(\zeta a)}$$

Note that the phase always cancels in the trace.

# Weak coupling

For weak coupling, a major simplification results because of the orthogonality property,

$$\sum_{m=-l}^l \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix} = \delta_{l''l''} \frac{1}{2l'' + 1}, l \leq l'.$$

$$E = -\frac{\lambda_1 a \lambda_2 b}{4R} \int_0^\infty \frac{dx}{x} \sum_{l''l''} (2l + 1)(2l' + 1)(2l'' + 1) \\ \times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix}^2 K_{l''+1/2}^2(x) I_{l+1/2}^2(xa/R) I_{l'+1/2}^2(xb/R).$$

# Power series expansion

As with the cylinders, we expand the modified Bessel functions of the first kind in power series in  $a/R, b/R < 1$ . This expansion yields the infinite series

$$E = -\frac{\lambda_1 a \lambda_2 b}{4\pi R} \frac{ab}{R^2} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{m=0}^n D_{n,m} \left(\frac{a}{R}\right)^{2(n-m)} \left(\frac{b}{R}\right)^{2m}$$

where by inspection of the first several  $D_{n,m}$  coefficients we can identify them as

$$D_{n,m} = \frac{1}{2} \binom{2n+2}{2m+1},$$

# Closed form

and now we can immediately sum the expression for the Casimir interaction energy to give the closed form

$$E = \frac{\lambda_1 a \lambda_2 b}{16\pi R} \ln \left( \frac{1 - \left(\frac{a+b}{R}\right)^2}{1 - \left(\frac{a-b}{R}\right)^2} \right) .$$

Again, when  $d = R - a - b \ll a, b$ , the proximity force theorem is reproduced:

$$U(d) \sim \frac{\lambda_1 \lambda_2 ab}{16\pi R} \ln(d/R), \quad d \ll a, b.$$

However, as the figures demonstrate, the approach is not very smooth, even for equal-sized spheres. The ratio of the energy to the PFA is ( $b/a = \mu$ )

$$\frac{E}{U} = 1 + \frac{\ln[(1 + \mu)^2 / 2\mu]}{\ln d/R}, \quad d \ll a, b.$$

# $a = b$ ; truncation at 100 shown

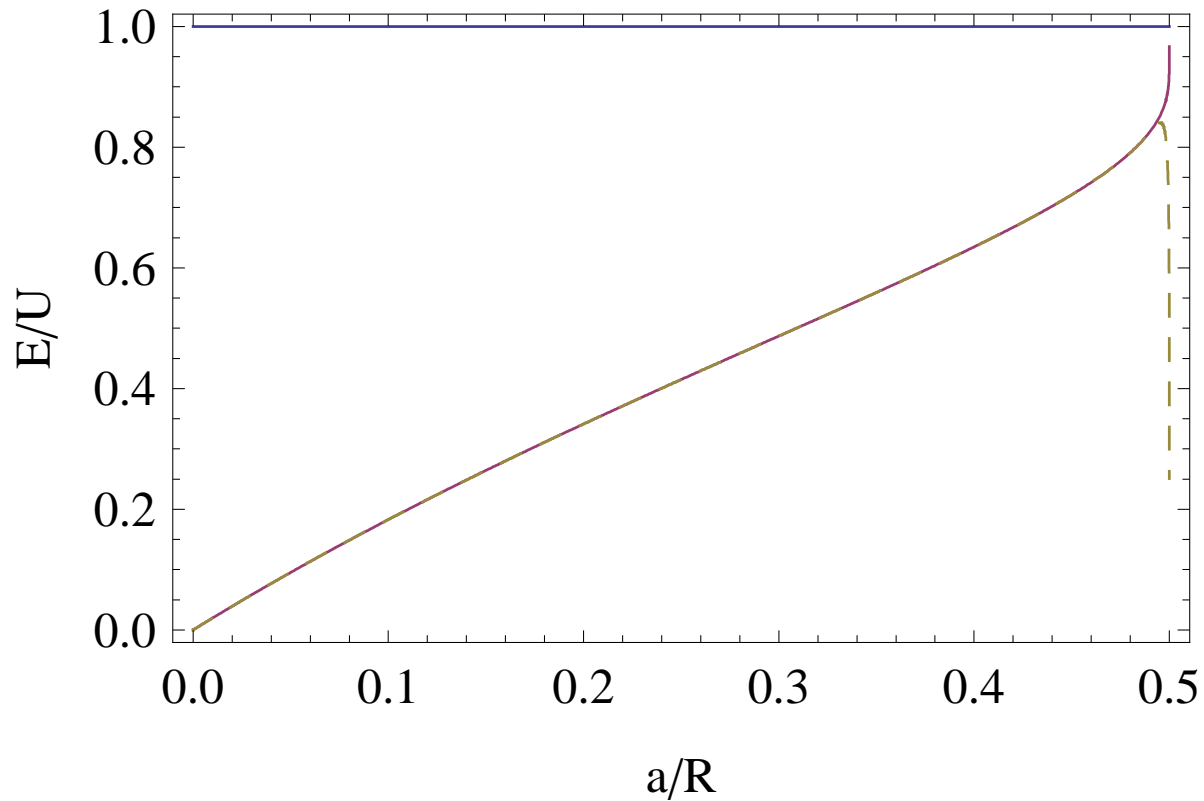


Figure 4: Plotted is the ratio of the exact interaction energy of two weakly-coupled spheres to the proximity force approximation

$$b/a = 49$$

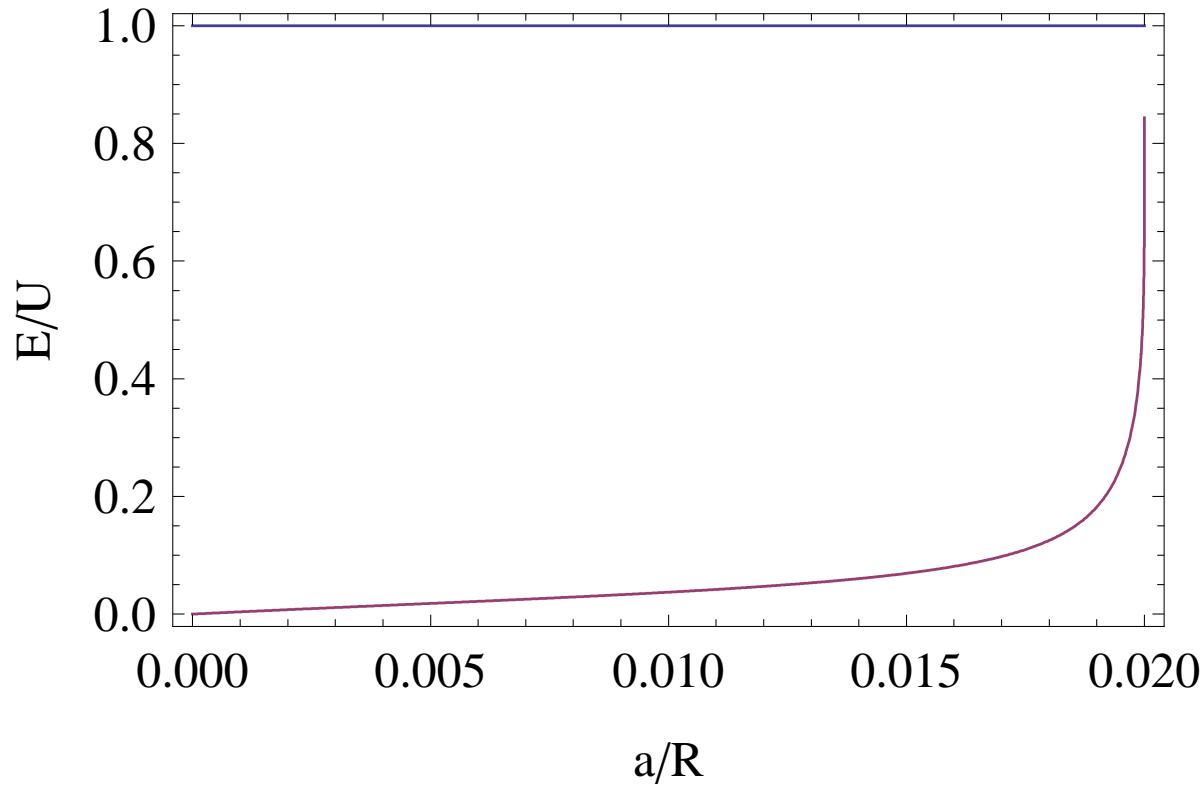


Figure 5: Plotted is the ratio of the exact interaction energy of two weakly-coupled spheres to the proximity force approximation

# Exact plane/sphere energy

In just the way indicated above, we can obtain a closed-form result for the interaction energy between a weakly-coupled sphere and a Dirichlet plane. Using the simplification that

$$\sum_{m=-l}^l (-1)^m \begin{pmatrix} l & l & l' \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} l & l & l' \\ 0 & 0 & 0 \end{pmatrix} = \delta_{l'0},$$

we can write the interaction energy in the form

$$E = -\frac{\lambda a}{2\pi R} \int_0^\infty dx \sum_{l=0}^{\infty} \sqrt{\frac{\pi}{2x}} (2l+1) K_{1/2}(x) I_{l+1/2}^2 \left( x \frac{a}{R} \right)$$

Then in terms of  $R/2$  as the distance between the center of the sphere and the plane, the exact interaction energy is

$$E = -\frac{\lambda}{2\pi} \left(\frac{a}{R}\right)^2 \frac{1}{1 - (2a/R)^2},$$

which as  $a \rightarrow R/2$  reproduces the proximity force limit, contained in the (ambiguously defined) PFA formula

$$U = -\frac{\lambda a}{8\pi d}.$$

# Exact energy vs. PFA

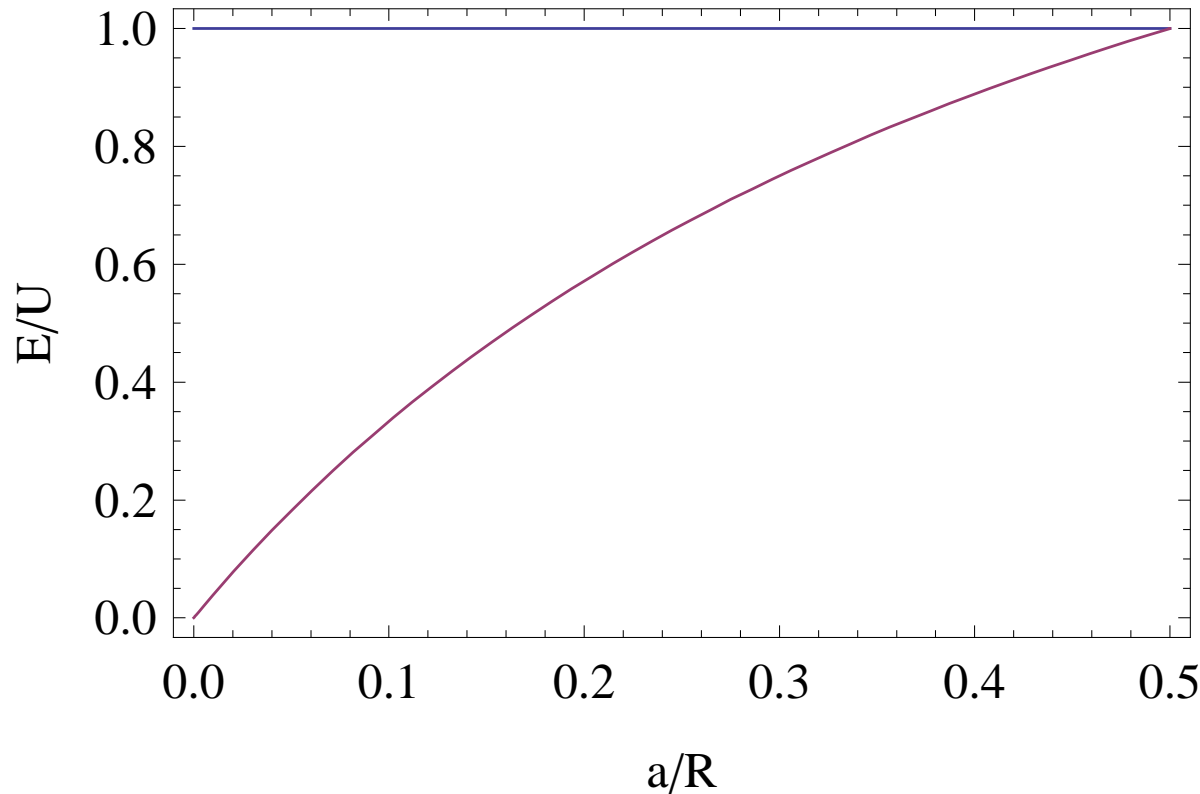


Figure 6: Plotted is the ratio of the exact interaction energy of a weakly-coupled sphere above a Dirichlet plane to the PFA.

# Comments and Prognosis

- The methods proposed are in fact not particularly novel, and illustrate the ability of physicists to continually rediscover old methods.
- What is new is the ability, partly due to enhancement in computing power and flexibility, to evaluate continuum determinants (or infinitely dimensional discrete ones) accurately numerically.
- This will make it possible to compute Casimir forces for geometries previously inaccessible.

# New results

- It is indeed remarkable, if perhaps not surprising in retrospect, to see that closed form expressions can be obtained for the interaction between spheres and between parallel cylinders in weak coupling.
- These results demonstrate most conclusively the unreliability of the proximity force approximation (of course, the proximity force theorem holds true).
- Further applications of our method will be given in the talks by Shajesh and Prachi. (See also Jef's talk.)