Casimir torque between corrugated surfaces: I. Next to leading order contributions.

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Abstract

1. We calculate the lateral Casimir force between corrugated parallel plates, described by δ-potentials, while interacting with a scalar field, using the multiple scattering formalism.
2. The contributions to the Casimir energy due to un-corrugated parallel plates is treated as a background from the outset.
3. We derive the leading, and next to leading, order contribution to the lateral Casimir force for the case when the mean of the corrugation amplitudes are small in comparison to the mean distance between the plates.
4. The analogous calculation for the electromagnetic case will decrease the error in the theoretical calculation sufficiently for comparison with experiments.

Outline

1. Motivation
   - Physical parameters
2. Formalism
   - Vacuum energy in field theory
   - Statement of the problem
   - Casimir energy contributing to the lateral force
3. Second order perturbation
   - Leading order contribution
   - Leading order contribution: Dirichlet limit
   - Sinusoidal corrugations: Leading order
4. Fourth order perturbation
   - Lateral Casimir Force
5. **Conclusions and Things to do**

Motivation

1997: Lateral force between two corrugated plates was studied by Golestanian et. al.. (Incidentally, this whole project was motivated after attending a talk by Golestanian, in Leipzig, where he talked about the rack and pinion arrangement.)
1999: Experimental study of lateral forces conducted by Mohideen et. al.. They studied the arrangement of a plate, with small sinusoidal corrugations, and a large sphere to study the nontrivial boundary dependence of the Casimir force.
2001: Emig et. al. evaluated analytic expressions for the lateral Casimir force between two corrugated parallel plates, in leading order, for the case when the corrugation amplitudes were small in comparison to the distance between the plates.
2002: The theoretical analysis of Mohideen's experimental data presented by Mostepanenko et. al. using PFA technique.
2007: Debate and controversy between Rodrigues et. al. and Mostepanenko et. al..

Experimental parameters

$$
\begin{aligned}
h &= 60 \text{ nm}, \\
d &= 1.1 \ \mu\text{m}, \\
a &= 0.1 - 0.9 \ \mu\text{m}.
\end{aligned}
$$

This corresponds to

$$
\begin{aligned}
k_0 a &= 0.6 - 5.1, \\
\frac{h}{a} &= 0.07 - 0.6, \\
\frac{h}{d} &= 0.05.
\end{aligned}
$$

Multiple scattering formalism

Let us consider a scalar field, $\phi(x)$, interacting with a scalar background field, $V(x)$, described by the Lagrangian density

$$\mathcal{L}(\phi(x)) = -\frac{1}{2}\,\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}\,V(x)\,\phi(x)^2.$$

Multiple scattering formalism: As described in the previous talk by Kim, the energy of the vacuum in the presence of the background field is given by the expression

$$E = \frac{i}{2\tau}\,\mathrm{Tr}\ln GG_0^{-1},$$

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where, the Green's function G is defined by the equation

$$-\left[\partial^2 - V(x)\right] G(x,x') = \delta^{(0)}(x-x'),$$

and G_0 is the corresponding Green's function when the background is switched off.

Statement of the problem

Corrugated, semi-transparent, parallel plates, are described by

$$V_i(z,y) = \lambda_i \, \delta(z - a_i - h_i(y)), \quad i = 1, 2, \quad a = a_2 - a_1 > 0,$$

where, the functions $h_i(y)$ describe the corrugations.

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where, the functions $h_i(y)$ describe the corrugations.
Total Casimir energy for a configuration when one of the plate is laterally shifted by an amount y_0,

$$h_1(y + y_0), \quad h_2(y),$$

can be written in the form

$$\Delta E(a, h_i, y_0) = E - E^{(0)}(a) = E_1(a, h_1) + E_2(a, h_2) + E_{12}(a, h_i, y_0),$$

where, E_{12}, isolates the interaction energy due to the lateral shift.

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Lateral Casimir force is defined as the negative change in energy due to the lateral shift:

$$F_{\|}(a, h_i, y_0) = -\frac{\partial}{\partial y_0}\Delta E = -\frac{\partial E_{12}}{\partial y_0}.$$

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Notation: Δ means deviation from the background, or a reference.

Casimir energy contributing to the lateral force

Using the multiple scattering formalism the contribution to energy due to the lateral shift is given by

$$\Delta E_{12} = -\frac{i}{2\tau}\,\mathrm{Tr}\ln\Big[1 - G_1 \Delta V_1 G_2 \Delta V_2\Big],$$

where,

$$\Delta V_i(z,y) = V_i(z,y) - V_i^{(0)}(z).$$

The Green's function associated with the background satisfies the differential equation,

$$(-\partial^2 + V_1^{(0)} + V_2^{(0)})G^{(0)} = 1.$$

and the differential equations for G_i's are

$$\Big[-\partial^2 + V_1^{(0))} + V_2^{(0)} + \Delta V_i\Big]\, G_i = 1.$$

Corrugations as a perturbation

In those cases when the corrugations can be treated as a small perturbation over the background, $(h_i \ll a)$, we can approximate

$$\Delta V_i^{(1)}(z,y) = -h_i(y)\,\frac{\partial}{\partial z} V_i^{(0)}(z) = -\lambda_i h_i(y) \frac{\partial}{\partial z} \delta(z-a_i) + \mathcal{O}(h/a)^2,$$

$$G_i \Delta V_i = \left[1 + G^{(0)}\Delta V_i\right]^{-1} G^{(0)} \Delta V_i = G^{(0)} \Delta V_i^{(1)} + \mathcal{O}(h/a)^2,$$

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$$G_i\Delta V_i = \Big[1 + G^{(0)}\Delta V_i\Big]^{-1} G^{(0)}\Delta V_i = G^{(0)}\Delta V_i^{(1)} + \mathcal{O}(h/a)^2,$$

Thus, to the leading order in the perturbation, the interaction energy will be

$$\Delta E_{12}^{(2)} = \frac{i}{2\tau}\,\mathrm{Tr}\Big[G^{(0)}\Delta V_1^{(1)} G^{(0)}\Delta V_2^{(1)}\Big],$$

where, $G^{(0)}$, are the Green's functions when the corrugations are switched off, which is already available to us in closed form from our earlier work.

Leading order contribution

The interaction energy can be written in the form

$$\frac{\Delta E_{12}^{(2)}}{L_x} = \int_{-\infty}^{+\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{+\infty} \frac{dk_2}{2\pi} \, \tilde{h}_1(k_1 - k_2) \, \tilde{h}_2(k_2 - k_1) \, L^{(2)}(k_1, k_2)$$

where $\tilde{h}_i(k)$ are the Fourier transforms of the functions $h_i(y)$, which describe the corrugations on the parallel plates,

$$\tilde{h}_i(k) = \int_{-\infty}^{+\infty} dy \, e^{-iky} \, h_i(y)$$

and the kernel $L^{(2)}(k_1, k_2)$ is suitably expressed in the form

$$L^{(2)}(k_1, k_2) = -\frac{1}{4\pi} \int_0^{\infty} \bar{\kappa} \, d\bar{\kappa} \; I^{(2)}(\kappa_1, \kappa_2),$$

where, $\kappa_i^2 = \bar{\kappa}^2 + k_i^2$, and

$$I^{(2)}(\kappa_1, \kappa_2) = \lambda_1 \lambda_2 \left[\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \, g^{(0)}(z, \bar{z}; \kappa_1) \, g^{(0)}(\bar{z}, z; \kappa_2) \right]_{\bar{z}=a_1, z=a_2} .$$

The $g^{(0)}$'s are available in closed form.

Leading order contribution (contd)

Using the reciprocal symmetry in the Green's function we learn

$$I^{(2)}(\kappa_1, \kappa_2) = I^{(2)}(\kappa_2, \kappa_1).$$

We evaluate the derivatives in the expression for $I^{(2)}(\kappa_1, \kappa_2)$, which will be described by Prachi Parashar in the corresponding calculation for studying Non Contact Gears, as

$$\begin{aligned} I^{(2)}(\kappa_1, \kappa_2) &= -\frac{\lambda_1}{2\kappa_1}\frac{\lambda_2}{2\kappa_2}\frac{e^{-a(\kappa_1+\kappa_2)}}{\Delta_1\Delta_2} \\ &\quad \left[\kappa_1^2\left(1+\frac{\lambda_1}{2\kappa_1}\right)\left(1+\frac{\lambda_2}{2\kappa_1}\right)+\kappa_1\kappa_2\left(1+\frac{\lambda_1}{2\kappa_1}\right)\left(1+\frac{\lambda_2}{2\kappa_2}\right)\right. \\ &\quad \left. +\kappa_1\kappa_2\left(1+\frac{\lambda_1}{2\kappa_2}\right)\left(1+\frac{\lambda_2}{2\kappa_1}\right)+\kappa_2^2\left(1+\frac{\lambda_1}{2\kappa_2}\right)\left(1+\frac{\lambda_2}{2\kappa_2}\right)\right] \end{aligned}$$

where

$$\Delta_i = 1 + \frac{\lambda_1}{2\kappa_i} + \frac{\lambda_1}{2\kappa_i} + \frac{\lambda_1}{2\kappa_i}\frac{\lambda_2}{2\kappa_i}\left(1 - e^{-2\kappa_i a}\right)$$

Leading order contribution: Dirichlet limit

For the case of Dirichlet limit ($\lambda_{1,2} \to \infty$) the expression for $I^{(2)}(\kappa_1, \kappa_2)$ takes the relatively simple form

$$I_D^{(2)}(\kappa_1, \kappa_2) = -\frac{\kappa_1}{\sinh \kappa_1 a} \frac{\kappa_2}{\sinh \kappa_2 a}$$

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where the subscript D stands for Dirichlet limit.
Using the above expression we can evaluate the interaction energy to be

$$\frac{\Delta E_{12}^{(2)}}{L_x} = \int_{-\infty}^{+\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{+\infty} \frac{dk_2}{2\pi} \, \tilde{h}_1(k_1 - k_2) \, \tilde{h}_2(k_2 - k_1) \, L_D^{(2)}(k_1, k_2)$$

where

$$L_D^{(2)}(k_1, k_2) = \frac{1}{4\pi} \int_0^{\infty} \bar{\kappa} \, d\bar{\kappa} \frac{\kappa_1}{\sinh \kappa_1 a} \frac{\kappa_2}{\sinh \kappa_2 a}.$$

Sinusoidal corrugations: Energy to leading order

For the particular case of sinusoidal corrugations we will have

$$h_1(y) = h_1 \sin[k_0(y + y_0)], \qquad h_2(y) = h_2 \sin[k_0 y],$$

where, $k_0 = 2\pi/d$ is the wave-number corresponding to the corrugation wavelength d. The Fourier transform is

$$\tilde{h}_1(k) = h_1 \frac{2\pi}{2i} \Big[e^{ik_0 y_0} \delta(k - k_0) - e^{-ik_0 y_0} \delta(k + k_0) \Big]$$

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which lets us write the expression for energy as

$$\frac{\Delta E_{12}^{(2)}}{L_x L_y} = \frac{\pi^2}{240\, a^3} \frac{h_1}{a} \frac{h_2}{a} A_D^{(1,1)}(k_0 a) \cos k_0 y_0,$$

where

$$A_D^{(1,1)}(t_0) = \frac{15}{\pi^4} \int_0^\infty \bar{s}\, d\bar{s} \int_{-\infty}^{+\infty} dt \frac{s}{\sinh s} \frac{s_+}{\sinh s_+}$$

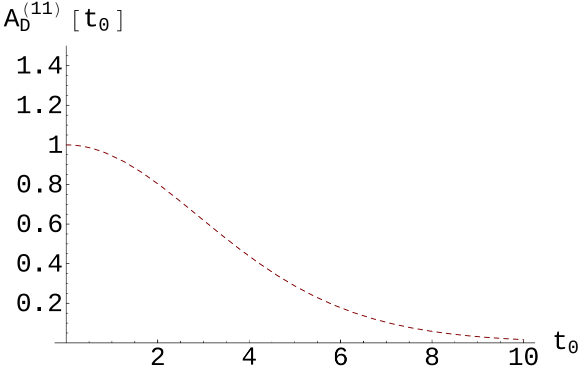
where, $s^2 = \bar{s}^2 + t^2$, and $s_\pm^2 = \bar{s}^2 + (t \pm t_0)^2$.

Sinusoidal corrugations: Force to leading order

The lateral Casimir force evaluates to

$$F_{\parallel}\left(k_0 a, \frac{h_i}{a}, k_0 y_0\right) = 2\,|F_{\perp}^{(0)}|\,\frac{h_1}{a}\frac{h_2}{a}\,k_0 a\,\sin(k_0 y_0)\,A_D^{(1,1)}(k_0 a),$$

where, $|F_{\perp}^{(0)}|$ is the magnitude of the normal Casimir force between two uncorrugated parallel plates.



Fourth order perturbation

The second correction to the lateral Casimir force is more laborious, but conceptually similar. It evaluates to

$$F_{\parallel}^{(4)}\left(k_0 a, \frac{h_i}{a}, k_0 y_0\right) = 2|F_{\perp}^{(0)}|\, k_0 a\, \sin(k_0 y_0)\, A_D^{(4)}(k_0 a)$$

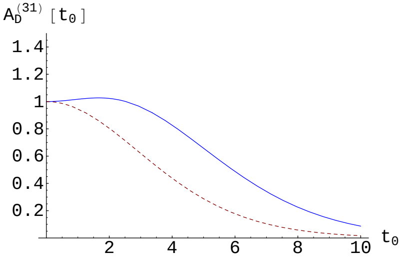
$$\begin{aligned} A_D^{(4)}(k_0 a) \;&=\; \frac{h_1}{a}\frac{h_2}{a}\frac{15}{4}\left[\frac{h_1^2}{a^2}\, A_D^{(3,1)}(k_0 a) + \frac{h_2^2}{a^2}\, A_D^{(1,3)}(k_0 a)\right. \\ &\qquad \left. -2\frac{h_1}{a}\frac{h_2}{a}\, \cos k_0 y_0\, A_D^{(2,2)}(k_0 a)\right]. \end{aligned}$$

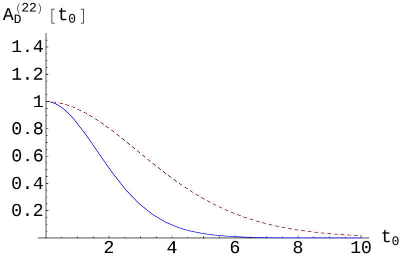
$$\begin{aligned} A_D^{(3,1)} \;&=\; \frac{1}{2\pi^4}\int_0^\infty \bar{s}\, d\bar{s} \int_{-\infty}^{+\infty} dt \frac{s}{\sinh s}\frac{s_+}{\sinh s_+}\left[4\frac{s}{\tanh s}\frac{s_-}{\tanh s_-}\right. \\ &\qquad \left. +2\frac{s}{\tanh s}\frac{s_+}{\tanh s_+} - s^2 - s_-^2\right] = A_D^{(1,3)}(t_0) \end{aligned}$$

$$A_D^{(2,2)} \;=\; \frac{1}{\pi^4}\int_0^\infty \bar{s}\, d\bar{s}\int_{-\infty}^{+\infty} dt \left[\frac{s^2}{\sinh^2 s}\frac{s_-^2}{\sinh^2 s_-} + 2\frac{s^2}{\tanh^2 s}\frac{s_+}{\sinh s_+}\frac{s_-}{\sinh s_-}\right]$$

Numerical plots

The following are the plots for $A^{(3,1)}(k_0 a)$ and $A^{(2,2)}(k_0 a)$.
The plot in red is $A^{(1,1)}(k_0 a)$.





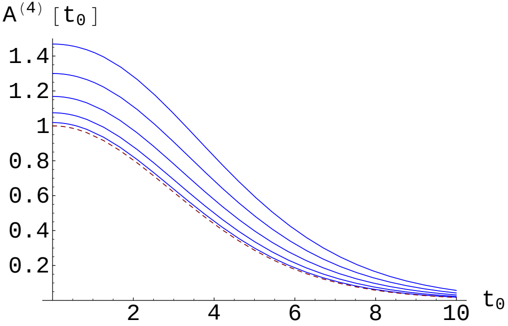
Note that

$$\begin{aligned} A^{(3,1)}(k_0 a) - A^{(1,1)}(k_0 a) &> 0 \\ A^{(2,2)}(k_0 a) - A^{(1,1)}(k_0 a) &< 0. \end{aligned}$$

Lateral Casimir Force

Let us now confine to the case: $h_1 = h_2$, and $k_0 y_0 = \frac{\pi}{2}$

$$F_{\|}\left(k_0 a, \frac{h}{a}, \frac{\pi}{2}\right) = 2 k_0 a \, |F_{\perp}^{(0)}| \frac{h^2}{a^2} \left[A_D^{(1,1)}(k_0 a) + \frac{15}{2} \frac{h^2}{a^2} A_D^{(3,1)}(k_0 a) \right]$$



$\frac{h}{a} = 0.05, 0.10, 0.15, 0.20, 0.25.$

Conclusions and Things to do

1. We have calculated the lateral Casimir force in the leading order, and in the next to leading order, when the corrugation amplitudes on parallel plates can be treated as a small perturbation relative to the distance between the plates.

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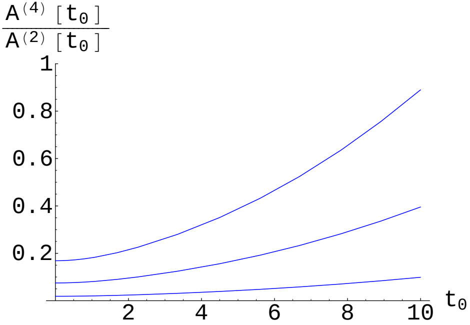
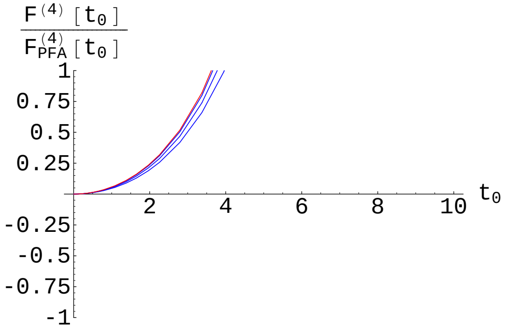
The talk was based on:

- Inés Cavero-Peláez, Kimball A. Milton, Prachi Parashar, K. V. Shajesh, Casimir torque between corrugated surfaces: I. Next to leading order contribution, *Under preparation*.
- Inés Cavero-Peláez, Kimball A. Milton, Prachi Parashar, K. V. Shajesh, Casimir torque between corrugated surfaces: II. Non contact gears, *Under preparation*.

$A^{(4)}(t_0)$ verses $A^{(2)}(t_0)$:



$F^{(4)}_{PER}(t_0)$ verses $F^{(4)}_{PFA}(t_0)$:



$F^{(4)}_{PER}(t_0)$ verses $F_{PFA}(t_0)$: