Distributional interpretation of Casimir energies

Linearized Einstein eq. (related to a particular path)

$$-\frac{d^2 h}{dx^2} = \frac{8}{\pi}\,\frac{4x^2-3t^2}{(t^2+4x^2)^3}\,\theta(x) \quad (1)$$ (h is the 00-component of the metric tensor)

integrating twice and imposing that h as well as its derivative vanish at $x=0$ [note the half-space $x<0$ is assumed to be empty]

$$\Rightarrow h(x) = \frac{\theta(x)}{\pi}\left[\frac{4x}{t^3}\tan^{-1}\left(\frac{2x}{t}\right) - \frac{1}{t^2+4x^2} + \frac{1}{t^2}\right]. \quad (2)$$

As $t\to 0$ which corresponds to removing the regularization parameter this expression is ill-defined in the usual sense of h being a fct. Therefore we consider a distributional interpretation of (2) and we check if eq. (1) remains valid as $t \to 0$ in distributional sense.

The basis for the analysis to come is the following theorem.

Moment expansion theorem: Let $f \in S'(\mathbb{R})$ with support bounded on the left. Suppose $f(x) = b_1 x^{\beta_1} + \dots + b_n x^{\beta_n} + O(x^{\beta})$ as $x\to\infty$, where $\beta_1 > \dots > \beta_n > \beta$, and $-(k+1) > \beta > -(k+2)$. Then as $\lambda \to \infty$ ($\hat{=}\ t\to 0$)

$$f(\lambda x) = \sum_{j=1}^{n} b_j\, g_j(\lambda x) + \sum_{j=0}^{k} (-1)^j \mu_j(f)\,\delta^{(j)}(\lambda x)/j! + O(\lambda^{\beta})$$

in the space $S'(\mathbb{R})$, where (*) $g_j(x) = Pf\left(x^{\beta_j}\theta(x)\right)$ if $\beta_j = -1,-2,-3,\dots$

Pf = Pseudo fct.

Here the moments are

$$\mu_j(f) = F.P.\int_{-\infty}^{\infty} f(x)\,x^j\,dx.$$

$F.P.$ = finite part

(*) $g_j(x) = x^{\beta_j}\theta(x)$ if $\beta_j \neq -1,-2,-3,\dots$ and

Jasson working on more general results; leave the proof to him.

Explanation of notation:

Let $F(\varepsilon) = \int_{\varepsilon}^{a} f(x)\,dx$ be given as $F(\varepsilon) = F_1(\varepsilon) + F_2(\varepsilon)$, where $F_1(\varepsilon)$ is singular as $\varepsilon \to 0$ and $\lim_{\varepsilon \to 0} F_2(\varepsilon) = A$ exists. We then define

$$F.P.\ F(\varepsilon) = A.$$

[Similarly for $F(M) = \int_0^M f(x)\,dx$ as $M \to \infty$.]

Definition of Pseudofunctions:

$$\langle Pf(g(x)), \phi(x) \rangle = F.P. \int_0^\infty g(x)\,\phi(x)\,dx.$$

Application of the Moment expansion theorem to $h(x)$, eq. (2) $\left(\lambda = \frac{1}{t}\right)$:

Write $h(x) = h_1(x) + h_2(x) + h_3(x)$ with

$$h_1(x) = \frac{\theta(x)}{\pi}\,\frac{4x}{t^3}\,\tan^{-1}\left(\frac{2x}{t}\right) = \frac{4}{\pi}\,\lambda^2 \cdot \theta(x)\,(\lambda x)\,\tan^{-1}(2\lambda x)$$

$$h_2(x) = -\frac{\theta(x)}{\pi}\,\frac{1}{t^2 + 4x^2} = -\frac{\lambda^2}{\pi}\,\theta(x)\,\frac{1}{1 + 4\lambda^2 x^2}$$

$$h_3(x) = \frac{\theta(x)}{\pi} \cdot \frac{1}{t^2} = \lambda^2 \cdot \frac{\theta(x)}{\pi}.$$

done

Asymptotic behavior of h_3: done.

Asymptotic behavior of h_2:

$$f_2(x) = \frac{\theta(x)}{1+4x^2} = \frac{1}{4x^2} + \mathcal{O}\left(\frac{1}{x^4}\right), \quad \text{so } -3 < \beta < -4 \text{ and}$$

$$h_2(x) = -\frac{\lambda^2}{\pi}\left\{\frac{1}{4}\,\mathrm{Pf}\left[\frac{\theta(\lambda x)}{(\lambda x)^2}\right] + \sum_{j=0}^{2} \frac{(-1)^j \mu_j(f_2)\,\delta^{(j)}(\lambda x)}{j!} + \mathcal{O}(\lambda^\beta)\right\}$$

($\mu_j(f_2)$: numbers; easily evaluated)

Asymptotic behavior of h_1:

$$f_1(x) = \theta(x)\, x \tan^{-1}(2x) = \frac{\pi}{2}\cdot x - \frac{1}{2} + \frac{1}{24}\cdot\frac{1}{x^2} + \mathcal{O}\left(\frac{1}{x^4}\right), \text{ so } -3<\beta<4 \text{ and}$$

$$h_1(x) = \frac{4}{\pi}\lambda^2\left\{\frac{\pi}{2}\theta(\lambda x)(\lambda x) - \frac{1}{2}\theta(\lambda x)(\lambda x)^0 + \frac{1}{24}\,\mathrm{Pf}\left[\frac{\theta(\lambda x)}{(\lambda x)^2}\right] + \sum_{j=0}^{2}\frac{(-1)^j \mu_j(f_1)\,\delta^{(j)}(\lambda x)}{j!} + \mathcal{O}(\lambda^\beta)\right\}$$

($\mu_j(f_1)$: numbers; easily evaluated)

In order to identify the $\lambda\to\infty$ behavior more clearly we need to understand scaling properties of distributions in some detail.

Def.: The distribution $g(Ax)$ is defined as

$$\langle g(Ax), \phi(x)\rangle = \frac{1}{|\det A|}\langle f(x), \phi(A^{-1}x)\rangle.$$

Pedantic point of view | terms in h_1

Ex.:

$$\langle \theta(\lambda x)(\lambda x), \phi(x)\rangle = \frac{1}{\lambda}\left\langle \theta(x)\cdot x, \phi\left(\frac{x}{\lambda}\right)\right\rangle = \frac{1}{\lambda}\int_0^\infty x\,\phi\left(\frac{x}{\lambda}\right)dx$$

(subst.: $y = \frac{x}{\lambda}$)

$$= \frac{1}{\lambda}\int_0^\infty \lambda y\,\phi(y)\,\lambda\,dy = \lambda\langle\theta(x)\cdot x, \phi(x)\rangle.$$

similarly for the second term

Ex.: $\langle \delta^{(j)}(\lambda x), \phi(x)\rangle = \frac{1}{\lambda}\langle \delta^{(j)}(x), \phi\left(\frac{x}{\lambda}\right)\rangle$

$$= \frac{(-1)^j}{\lambda}\langle \delta(x), \frac{d^j}{dx^j}\phi\left(\frac{x}{\lambda}\right)\rangle = \frac{(-1)^j}{\lambda}\left[\frac{d^j}{dx^j}\phi\left(\frac{x}{\lambda}\right)\right]\Big|_{x=0}$$

$$= \frac{(-1)^j}{\lambda^{j+1}}\phi^{(j)}(0) = \frac{(-1)^j}{\lambda^{j+1}}\langle \delta(x), \phi^{(j)}(x)\rangle = \frac{1}{\lambda^{j+1}}\langle \delta^{(j)}(x), \phi(x)\rangle.$$

Ex. pseudo fct.: Before we consider the scaling we derive some properties of the pseudo-fct.

Lemma: $\langle Pf\left(\frac{\theta(x)}{x}\right), \phi(x)\rangle = \int_0^1 \frac{\phi(x)-\phi(0)}{x}dx + \int_1^\infty \frac{\phi(x)}{x}dx$

Proof: Note, that [by def. of Pf this is what we have to consider]

$$\int_\varepsilon^\infty \frac{\phi(x)}{x}dx = \left\{\int_\varepsilon^1 + \int_1^\infty\right\}\frac{\phi(x)}{x}dx = \int_\varepsilon^1 \frac{(\phi(x)-\phi(0)) + \phi(0)}{x}dx + \int_1^\infty \frac{\phi(x)}{x}dx$$

($\frac{\phi(x)-\phi(0)}{x}$: $x\to 0$ behavior improved)

$$= -\ln\varepsilon\,\phi(0) + \int_\varepsilon^1 \frac{\phi(x)-\phi(0)}{x}dx + \int_1^\infty \frac{\phi(x)}{x}dx. \blacksquare$$

This implies the assertion.

Lemma: $Pf\left[\frac{\theta(\lambda x)}{\lambda x}\right] = \frac{1}{\lambda}Pf\left[\frac{\theta(x)}{x}\right] + \frac{\ln\lambda\cdot\delta(x)}{\lambda}$ ← It's basically about understanding this term.

Proof: By def.

$$\langle Pf\left[\frac{\theta(\lambda x)}{\lambda x}\right], \phi(x)\rangle = \frac{1}{\lambda}\langle Pf\left[\frac{\theta(x)}{x}\right], \phi\left(\frac{x}{\lambda}\right)\rangle = \frac{1}{\lambda}\,F.P.\int_0^\infty \frac{\phi\left(\frac{x}{\lambda}\right)}{x}dx.$$

Clearly, [this is what needs to be considered]

$$\int_{\varepsilon}^{\infty} \frac{\phi\left(\frac{x}{\lambda}\right)}{x}\,dx = \left(\int_{\varepsilon}^{\lambda} + \int_{\lambda}^{\infty}\right) \frac{\phi\left(\frac{x}{\lambda}\right)}{x}\,dx$$

(reasonable splitting for scaling to come)

Furthermore,

$$\int_{\lambda}^{\infty} \frac{\phi\left(\frac{x}{\lambda}\right)}{x}\,dx \overset{y=\frac{x}{\lambda}}{=} \int_{1}^{\infty} \frac{\phi(y)}{y}\,dy$$

[fits the second term in previous lemma; will be part of first term in this lemma]

$$\int_{\varepsilon}^{\lambda} \frac{\phi\left(\frac{x}{\lambda}\right)}{x}\,dx = \int_{\varepsilon/\lambda}^{1} \frac{\phi(y)}{y}\,dy = \int_{\varepsilon/\lambda}^{1} \frac{\phi(y)-\phi(0)+\phi(0)}{y}\,dy$$

$$= -\ln(\varepsilon/\lambda)\,\phi(0) + \int_{\varepsilon/\lambda}^{1} \frac{\phi(y)-\phi(0)}{y}\,dy.$$

Using the definition and the previous lemma implies the assertion. ∎

Similarly for $Pf\left[\frac{\theta(x)}{x^2}\right]$; proof e.g. by induction or by direct calculation.

<u>Adding up:</u>

$$h(x) = 2\lambda^3\theta(x)\cdot x - \frac{\lambda^2}{\pi}\theta(x) - \frac{1}{12\pi} Pf\left[\frac{\theta(x)}{x^2}\right] \tag{3}$$

$$- \frac{1}{18\pi}\delta'(x) + \frac{1}{12\pi}\ln(2\lambda)\,\delta'(x) + \dots$$

(leave on board)

(plus terms that vanish as $\lambda \to \infty$)

Check eq. (1):

(rewrite in form suitable for $\lambda \to \infty$)

$$-\frac{d^2 h}{dx^2} = \frac{8}{\pi}\frac{4x^2-3t^2}{(t^2+4x^2)^3}\theta(x) = \frac{8}{\pi}\frac{t^2}{t^6}\frac{4\left(\frac{x}{t}\right)^2-3}{\left(1+4\left(\frac{x}{t}\right)^2\right)^3}\theta(x)$$

$$= \frac{8}{\pi}\lambda^4 \frac{4\lambda^2x^2-3}{(1+4\lambda^2x^2)^3}\theta(x)$$

[exactly as before; as $x\to\infty$ this behave like $1/x^4$ and the following is apparent]

I do not want to be as explicit as before, but the relevant $f(x)$ is

$$\left[\ f(x) = \theta(x)\frac{4x^2-3}{(1+4x^2)^3} \sim \frac{1}{16x^4} + \mathcal{O}\left(\frac{1}{x^6}\right);\quad -5>\beta>-6\ \right]$$

($j=4$ neglected as vanishes as $\lambda\to\infty$)

$$= \frac{8}{\pi}\lambda^4\left\{\frac{1}{16}\,\mathrm{Pf}\left(\frac{\theta(\lambda x)}{(\lambda x)^4}\right) + \sum_{j=0}^{3}(-1)^j \mu_j(f)\frac{\delta^{(j)}(\lambda x)}{j!} + \mathcal{O}\left(\frac{1}{\lambda^5}\right)\right\}$$

(apply derived scaling behavior)

$$= \frac{1}{2\pi}\,\mathrm{Pf}\left(\frac{\theta(x)}{x^4}\right) - 2\lambda^3\delta(x) + \frac{1}{\pi}\lambda^2\delta'(x) + \frac{1}{8\pi}\delta'''(x)$$

$$-\frac{1}{12\pi}\ln(2\lambda)\,\delta'''(x) + \mathcal{O}\left(\frac{1}{\lambda}\right).$$

This is the rhs! Evaluate lhs.

Evaluate $\frac{d^2h}{dx^2}$: (go back to eq. (3) $\longrightarrow$ 2nd, 4th and 5th terms are trivial; 2nd and 5th term already match; 4th not yet and more contributions need to generate δ''' terms)

Lets see how to evaluate derivatives of distributions!

Ex.: $\left\langle \frac{d^2}{dx^2}\theta(x)\cdot x, \phi(x)\right\rangle = \int_0^\infty x\,\frac{d^2\phi}{dx^2}\,dx = x\cdot\phi'(x)\Big|_0^\infty - \int_0^\infty \frac{d\phi}{dx}\,dx$ (partial integration)

$= \phi'(0) = \langle \delta(x), \phi(x)\rangle$. (first term in (3) matches 2nd above)

Ex.:
$$\frac{d}{dx}\,\mathrm{Pf}\left(\frac{\theta(x)}{x^k}\right) = -k\,\mathrm{Pf}\left(\frac{\theta(x)}{x^{k+1}}\right) + \frac{(-1)^k\,\delta^{(k)}(x)}{k!}$$

(expected) (needs explanation)

Proof: Note, that

$\left\langle \frac{d}{dx}\,\mathrm{Pf}\left(\frac{\theta(x)}{x^k}\right), \phi(x)\right\rangle = -\left\langle \mathrm{Pf}\left(\frac{\theta(x)}{x^k}\right),\right.$ [redacted] $\left.\phi'(x)\right\rangle.$

Consider: (partial integration)

$$\int_\varepsilon^\infty \frac{\phi'(x)}{x^k}\,dx = \frac{\phi(x)}{x^k}\Bigg|_\varepsilon^\infty - \int_\varepsilon^\infty \left(\frac{d}{dx}\frac{1}{x^k}\right)\phi(x)\,dx$$

$$= -\frac{\phi(\varepsilon)}{\varepsilon^k} - k\int_\varepsilon^\infty \frac{\phi(x)}{x^{k+1}}\,dx$$

(Taylor series exp.)

$$= -\frac{1}{\varepsilon^k}\sum_{j=0}^\infty \frac{\phi^{(j)}(0)}{j!}\,\varepsilon^j - k\int_\varepsilon^\infty \frac{\phi(x)}{x^{k+1}}\,dx.$$

Taking the F.P. implies the assertion. ∎

So:
$$-\frac{1}{12\pi}\,\frac{d^2}{dx^2}\,\mathrm{Pf}\left(\frac{\theta(x)}{x^2}\right) = -\frac{1}{2\pi}\,\mathrm{P.f.}\left(\frac{\theta(x)}{x^4}\right) - \frac{5}{72\pi}\,\delta'''(x).$$

Adding up all terms in $-\frac{d^2h}{dx^2}$, eq. (1) follows as $\lambda \to \infty$.

In particular, the finite part as $\lambda \to \infty$ satisfies the differential equation. So eventually defining the metric as the finite part of the distributional answer makes sense!

Proof of Moment exp. theorem: [redacted]

It is possible to find a constant M such that we can write $f = f_0 + f_1$, where f_0 has compact support and where $f_1(x)$ has support in $[0,\infty)$ and with $\left| f_1(x) - \sum_{j=1}^{n} b_j x^{\beta_j} \right| \leq M x^{\beta}$, $x \geq 0$.

> This is clearly possible for $x \geq 1$ from the asymptotic behavior. By suitably choosing f_0 it can be enforced also for $0 \leq x < 1$.

Then
$$f_0(\lambda x) = \sum_{j=0}^{k} \frac{(-1)^j \mu_j(f_0)\, \delta^{(j)}(x)}{j!\, \lambda^{j+1}} + O\left(\lambda^{-(k+2)}\right), \text{ as } \lambda \to \infty$$

Furthermore:

> give me a second to define this [$\mu_j(f_1)$]

$$\left\langle f_1(\lambda x) - \sum_{j=1}^{n} b_j g_j(\lambda x) - \sum_{j=0}^{k} \frac{(-1)^j \mu_j(f_1)\, \delta^{(j)}(x)}{j!\, \lambda^{j+1}},\ \phi(x) \right\rangle$$

$$= \left\langle f_1(\lambda x) - \sum_{j=1}^{n} b_j g_j(\lambda x),\ \phi(x) \right\rangle - \sum_{j=0}^{k} \frac{\phi^{(j)}(0)}{j!\, \lambda^{j+1}} \int_0^{\infty} \left(f_1(x) - \sum_{j=1}^{n} b_j g_j(x) \right) x^j \, dx$$

> well def. quantity by construction

> can be replaced by $x^{\beta_j} \lambda^{\beta_j}$ as integral is finite.

$$= \int_0^{\infty} \left(f_1(\lambda x) - \sum_{j=1}^{n} b_j g_j(\lambda x) \right) \left(\phi(x) - \sum_{j=0}^{k} \frac{\phi^{(j)}(0)}{j!} x^j \right) dx$$

> $\leftarrow O(x^{k+1})$ as $x \to 0$
>
> $\leftarrow O(x^{k})$ as $x \to \infty$

$$\leq \int_0^1 M x^{\beta} \lambda^{\beta} k x^{k+1} dx + \int_1^{\infty} M x^{\beta} \lambda^{\beta} k x^{k} dx = Mk \left[\underbrace{\frac{1}{k+\beta+2}}_{>0} - \underbrace{\frac{1}{k+\beta+1}}_{<0} \right] \lambda^{\beta}$$

$= O(\lambda^{\beta})$, implying the assertion. ■