The structure of quasiasymptotics of Schwartz distributions

Jasson Vindas

jvindas@math.lsu.edu

Louisiana State University

Quantum Vacuum Meeting

University of Oklahoma

March 15, 2008

Summary

The aim of this talk is to communicate new structural theorems for quasiasymptotics of Schwartz distributions.

- Review of the definition of quasiasymptotics at infinity and at the origin and the known properties.
- Integration of the quasiasymptotic and relationship with asymptotically and associate asymptotically homogeneous functions.
- Structural Theorems for quasiasymptotics at infinity and at the origin.
- The related question of extension of distributions.
- Particular case: the quasiasymptotic of order -1 at infinity.
- **Consequence**: Characterization of jump behavior of Fourier series in terms of Cesàro summability.
- **Consequence**: Pointwise Fourier inversion formula.

About the quasiasymptotic

The concept of the quasiasymptotic of Schwartz distributions was introduced in 1973 by B. I. Zavialov in

Automodel Asymptotics of Electromagnetic Form Factors in the Neighborhood of the Light Cone, Teoret. Mat. Fiz. 17, 178–188.

It resulted very useful in several areas such as quantum field theory and Tauberian theory.

Main results and applications are summarized in the book by Vladimirov, Drozhzhinov and Zavialov:

Tauberian Theorems for Generalized Functions (1988).

Notation

- $\mathcal{D}$ and $\mathcal{D}'$ denote the Schwartz spaces of test functions and distributions.
- $\mathcal{S}$ and $\mathcal{S}'$ are the spaces of rapidly decreasing functions and the space of tempered distributions.
- All of our functions and distributions are over the real line.
- The Fourier transform is defined as $\hat{\phi}(x) = \int_{-\infty}^{\infty} \phi(t) e^{ixt} \mathrm{d}t$.
- The evaluation of a distribution f at a test function ϕ is denoted by $\langle f(x), \phi(x) \rangle$
- Recall that the homogeneous distributions are x_-^{α}, x_+^{α}, if $\alpha \notin \mathbb{Z}_-$, and x^{-k}, $\delta^{k-1}(x)$, for negative integers.
- $H(x)$ is the Heaviside function.

The Basic Idea: Asymptotic Separation of Variables

We look for asymptotic representations of the form

$$f(\lambda x) \sim \rho(\lambda) g(x).$$

as either $\lambda \to \infty$ or $\lambda \to 0$, where g is a distribution and ρ is a positive measurable function.

The Basic Idea: Asymptotic Separation of Variables

We look for asymptotic representations of the form

$$f(\lambda x) \sim \rho(\lambda) g(x).$$

as either $\lambda \to \infty$ or $\lambda \to 0$, where g is a distribution and ρ is a positive measurable function.

If g is assumed to be different from 0, one can show that g and ρ must satisfy:

- $\rho(\lambda) = \lambda^{\alpha} L(\lambda)$, where L is a slowly varying function and α is called the index of variation.
- g is a homogeneous distribution of degree α.

Slowly Varying Functions

Recall that real-valued measurable function defined in some interval of the form $[A, \infty)$, $A > 0$, is called *slowly varying function at infinity* if L is positive for large arguments and

$$\lim_{x \to \infty} \frac{L(ax)}{L(x)} = 1,$$

for each $a > 0$.
Similarly one defines slowly varying functions at the origin.

Quasiasymptotic at infinity

Let L be slowly varying. We say that $f \in \mathcal{D}'$ has *quasiasymptotic behavior at infinity in* $\mathcal{D}'$ *with respect to* $\lambda^\alpha L(\lambda)$, $\alpha \in \mathbb{R}$, if for some $g \in \mathcal{D}'$ and every $\phi \in \mathcal{D}$,

$$\lim_{\lambda \to \infty} \left\langle \frac{f(\lambda x)}{\lambda^\alpha L(\lambda)}, \phi(x) \right\rangle = \langle g(x), \phi(x) \rangle \, .$$

We also say that f has *quasiasymptotic of order* α *at infinity with respect to* L.
We also express this by

$$f(\lambda x) = \lambda^\alpha L(\lambda) g(x) + o(\lambda^\alpha L(\lambda)), \ \lambda \to \infty \ \text{ in } \mathcal{D}'.$$

We may also have

$$f(\lambda x) = \lambda^\alpha L(\lambda) g(x) + o(\lambda^\alpha L(\lambda)), \ \lambda \to \infty \ \text{ in } \mathcal{S}'.$$

Quasiasymptotic at the Origin

Similarly, one defines the quasiasymptotic in $\mathcal{D}'$ and $\mathcal{S}'$ at the origin.

- By shifting, one can define the quasiasymptotic of distributions at any point.
- For example, Łojasiewicz defined the value of a distribution $f \in \mathcal{D}'$ at the point x_0 as the limit

  $$f(x_0) = \lim_{\varepsilon \to 0} f(x_0 + \varepsilon x)\,,$$

  if the limit exists in $\mathcal{D}'$.
- **Notation:** If $f \in \mathcal{D}'$ has a value γ at x_0, we say that $f(x_0) = \gamma$ in $\mathcal{D}'$. The meaning of $f(x_0) = \gamma$ in $\mathcal{S}'$, ..., must be clear.

Previous known properties at infinity

- If $f \in \mathcal{D}'$ has quasiasymptotic at infinity in $\mathcal{D}'$. Then, $f \in \mathcal{S}'$.
- Structural theorems when the order of the quasiasymptotic $\alpha \notin -\mathbb{N}$.
- If f has quasiasymptotic in $\mathcal{D}'$ whose order is not a negative integer, then f has the same quasiasymptotic in $\mathcal{S}'$. For $\alpha \in -\mathbb{N}$ the result was known only under the assumption L bounded.

Previous known properties at the origin

- Structural Theorem for $\alpha > 0$.
- Structural Theorem for $\alpha \in (-1, 0]$ under the assumption L bounded.
- If $f \in \mathcal{S}'$ has quasiasymptotic at the origin in $\mathcal{D}'$, then it has the same quasiasymptotic in $\mathcal{S}'$ in the following two cases,
  - $\alpha \leq 0$ and $\alpha \notin -\mathbb{N}$.
  - $\alpha > 0$ and L bounded.

Integration of the Quasiasymptotic

Suppose

$$f(\lambda x) = L(\lambda) g(\lambda x) + o(\lambda^{\alpha} L(\lambda)), \text{ in } \mathcal{D}',$$

(here $\lambda \to \infty$ or 0). Suppose that g admits a primitive G_k of order k which is homogeneous of degree $k + \alpha$. Then, for any given k-primitive F_k of f, there exist functions $b_0, \ldots, b_{k-1}$, such that

$$F_k(\lambda x) = L(\lambda) G_k(\lambda x) + \sum_{j=0}^{k-1} \lambda^{\alpha+k} b_j(\lambda) \frac{x^{k-1-j}}{(k-1-j)!} + o\left(\lambda^{\alpha+k} L(\lambda)\right), \text{ in } \mathcal{D}'$$

Integration of the Quasiasymptotic

Suppose

$$f\left(\lambda x\right)=L(\lambda)g(\lambda x)+o\left(\lambda^{\alpha}L(\lambda)\right),\ \text{in }\mathcal{D}',$$

(here $\lambda\to\infty$ or 0). Suppose that g admits a primitive G_k of order k which is homogeneous of degree $k+\alpha$. Then, for any given k-primitive F_k of f, there exist functions $b_0,\dots,b_{k-1}$, such that

$$F_k\left(\lambda x\right)=L(\lambda)G_k(\lambda x)+\sum_{j=0}^{k-1}\lambda^{\alpha+k}b_j(\lambda)\frac{x^{k-1-j}}{(k-1-j)!}+o\left(\lambda^{\alpha+k}L(\lambda)\right),\ \text{in }\mathcal{D}'$$

where

$$b_j(a\lambda)=a^{-\alpha-j-1}b_j(\lambda)+o(L(\lambda)).$$

Asymptotically Homogeneous Functions

Definition A function b is called **asymptotically homogeneous of degree α at infinity** (resp. at 0) if

$$b(ax) = a^{\alpha} b(x) + o(L(x)).$$

Properties

- In the case at infinity when $\alpha < 0$, or at 0 when $\alpha > 0$,

$$b(x) = o(L(x)).$$

- In the case at infinity when $\alpha > 0$, or at 0 when $\alpha < 0$,

$$b(x) = \beta x^{\alpha} + o(L(x)).$$

Structural Theorem for Some Cases

Theorem 1 *Let $f \in \mathcal{D}'$ have quasiasymptotic behavior at infinity (resp. at the origin) in $\mathcal{D}'$*

(1) $$f(\lambda x) = C_- L(\lambda) \frac{(\lambda x)_-^\alpha}{\Gamma(\alpha+1)} + C_+ L(\lambda) \frac{(\lambda x)_+^\alpha}{\Gamma(\alpha+1)} + o\left(\lambda^\alpha L(\lambda)\right).$$

If $\alpha \notin \{-1, -2, \ldots\}$, then there exist a positive integer m, an m-primitive F of f such that F is continuous (resp. continuous near the origin) and

(2) $$\lim \frac{\Gamma(\alpha+m+1) F(x)}{x^m |x|^\alpha L(|x|)} = C \pm .$$

Conversely, if these conditions hold, then (by differentiation) (1) follows.

Associate Asymptotically Homogeneous Functions

In the case of negative integer order, the main coefficient of integration of the quasiasymptotic satisfies the following definition.

Definition A function b is called **associate asymptotically homogeneous of degree 0 at infinity** (resp. at 0) with respect to L if

$$b(ax) = b(x) + \beta L(x) \log a + o(L(x)).$$

Structural Theorem for the Other Cases

Theorem 2 *f has the quasiasymptotic behavior in $\mathcal{D}'$ at infinity (resp. at the origin),*

$$f(\lambda x) = \gamma\lambda^{-k}L(\lambda)\delta^{(k-1)}(x) + (-1)^{k-1}\beta(k-1)!\lambda^{-k}L(\lambda)x^{-k} + o\left(\lambda^{-k}L(\lambda)\right)$$

if and only if there exist $m \in \mathbb{N}, m \geq k$, a function b satisfying $b(a\lambda) = b(\lambda) + \beta \log a L(\lambda) + o(L(\lambda))$ and a m-primitive F, which is continuous (resp. continuous near 0), such that

$$F(x) = b\left(|x|\right)\frac{x^{m-k}}{(m-k)!} + \gamma L\left(|x|\right)\frac{x^{m-k}}{2(m-k)!}sgn x$$

$$-\beta L\left(|x|\right)\frac{x^{m-k}}{(m-k)!}\sum_{j=1}^{m-k}\frac{1}{j} + o\left(|x|^{m-k}L\left(|x|\right)\right)$$

Second version of the Structural Theorem

This is a version free of b

Theorem 3 *Let $f \in \mathcal{D}'$. Then f has quasiasymptotic at infinity (resp. at the origin) of order $-k$, $k \in \{1, 2, ...\}$, if and only if there exists a continuous m-primitive F of f (resp. continuous near 0), $m > k$, such that for each $a > 0$,*

$$\lim_{x\to\infty} \frac{(m-k)!\left(a^{k-m}F(ax) - (-1)^{m-k}F(-x)\right)}{x^{m-k}L(x)} = I(a). \tag{3}$$

In such case I has the form $I(a) = \gamma + \beta \log a$.

The problem of regularization of distributions

Let f_0 be a distribution only defined on $\mathbb{R} \setminus \{0\}$.

Regularization deals with the problem of finding an extension of f_0 to $\mathbb{R}$. That is of importance in problems of renormalization in quantum field theories.

On can show that f_0 has an extension iff

$$f_0(\lambda x) = O(\lambda^\alpha), \;\; as\; \lambda \to 0^+ \;\; in\; \mathcal{D}'(\mathbb{R} \setminus \{0\})$$

In general there is no cannonical way to find the extension.

The quasiasymptotic could help in this problem: If f_0 has quasiasymptotic of order $\alpha \notin \mathbb{Z}_-$, then one can show using the structure that there is a regularization f having the same quasiasymptotic, being in general unique up to n constants ($n < -\alpha$).

A Particular Case: Order -1

Recall the definition of Cesaro limits of distributions. Let $g \in \mathcal{D}'$,

$$\lim_{x \to \infty} g(x) = \eta \; (\mathrm{C}, k),$$

if there exists a k-primitive G of g, being a regular distribution, such that $G(x) = \eta x^k/k! + o(x^k)$, as $x \to \infty$. Then the structural theorem for the quasiasymptotic of order -1 is the following:

$$f(\lambda x) = \gamma \delta(\lambda x) + \beta \, p.v. \left(\frac{1}{\lambda x} \right) + o\left(\frac{1}{\lambda} \right) \text{ as } \lambda \to \infty$$

if and only there is $k \in \mathbb{N}$ such that for all 1-primitive F of f and $a > 0$

$$\lim_{x \to \infty} F(ax) - F(-x) = \gamma + \beta \log a \; (\mathrm{C}, k).$$

First Consequence: Local behavior of Fourier Series

f is said to have a **jump behavior** at x_0 if

$$f(x_0 + \epsilon x) = \gamma_- H(-x) + \gamma_+ H(x) + o(1) \text{ in } \mathcal{D}' \text{ as } \epsilon \to 0^+.$$

Suppose that $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$, then it has this jump behavior at x_0 if and only if there exists $k \in \mathbb{N}$ such that for each $a > 0$

$$\lim_{x\to\infty} \sum_{-x \le n \le ax} a_n e^{inx_0} = \frac{\gamma_+ + \gamma_-}{2} + \frac{i}{2\pi}(\gamma_+ - \gamma_-)\log a \quad (\mathrm{C}, k).$$

e.v. Cesàro evaluations

Definition 1 *Let $g \in \mathcal{D}'$, and $k \in \mathbb{N}$. We say that the evaluation $\langle g(x), \phi(x) \rangle$ exists in the* $\mathrm{e.v.}$ *Cesàro sense, and write*

(4) $$\mathrm{e.v.}\, \langle g(x), \phi(x) \rangle = \gamma \quad (\mathrm{C}, k),$$

if for some primitive G of $g\phi$ and $\forall a > 0$ we have

$$\lim_{x \to \infty} (G(ax) - G(-x)) = \gamma \ (\mathrm{C}, k).$$

If g is locally integrable then we write (4) as

$$\mathrm{e.v.} \int_{-\infty}^{\infty} g(x)\, \phi(x)\, \mathrm{d}x = \gamma \quad (\mathrm{C}, k).$$

Remark: *In this definition the evaluation of g at ϕ does not have to be defined, we only require that $g\phi$ is well defined.*

Pointwise Fourier Inversion Formula

Now, we characterize the point values of a distribution in $\mathcal{S}'$ by using Fourier transform.

Theorem 4 *Let* $f \in \mathcal{S}'$*. We have* $f(x_0) = \gamma$ *in* $\mathcal{S}'$ *if and only if there exists a* $k \in \mathbb{N}$ *such that*

$$\frac{1}{2\pi}\text{e.v.}\left\langle \hat{f}\left(t\right), e^{-ix_0 t}\right\rangle = \gamma \;\; (\mathrm{C}, k)\,,$$

which in case $\hat{f}$ *is locally integrable means that*

$$\frac{1}{2\pi}\text{e.v.}\int_{-\infty}^{\infty}\hat{f}(t)e^{-ix_0 t}\mathrm{d}t = \gamma \;\; (\mathrm{C}, k)\,.$$

Open problem

The structure of the quasiasymptotic in several variables is still an open question.

The only known case is due to Vladimirov, Drozhzhinov and Zavialov for the quasiasymptotic at infinity of distributions supported on acute convex cones.