Recent Controversies in the Casimir Effect: Temperature, Entropy, and Self-Energies

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Outline

1. Introductory remarks

2. Dimensional dependence

   (a) Parallel plates in D spatial dimensions

   (b) Hyperspheres in D spatial dimensions

   (c) Cylinders in three dimensions

3. Connection with van der Waals forces

4. Temperature Dependence of the Casimir Effect

5. Divergences and Renormalization

6. Conclusion

Introduction

Zero-point fluctuations in quantum fields give rise to observable forces between material bodies, the so-called Casimir forces. In this lecture I present some results of the theory of the Casimir effect, primarily formulated in terms of Green's functions. There is an intimate relation between the Casimir effect and van der Waals forces. Applications to conductors and dielectric bodies of various shapes will be given for the cases of scalar, electromagnetic, and fermionic fields. The dimensional dependence of the effect will be described. However, real materials may not be well described by ideal boundary conditions; we will discuss the temperature dependence of the Casimir force between real metals, and whether the divergences that occur in calculation of self-stresses on bodies can be properly removed by renormalization.

We may identify the zero-point energy with the vacuum expectation value of the field energy,

$$\frac{1}{2}\sum_a \hbar\omega_a = \int (d\mathbf{x})\langle T^{00}(\mathbf{x})\rangle.$$

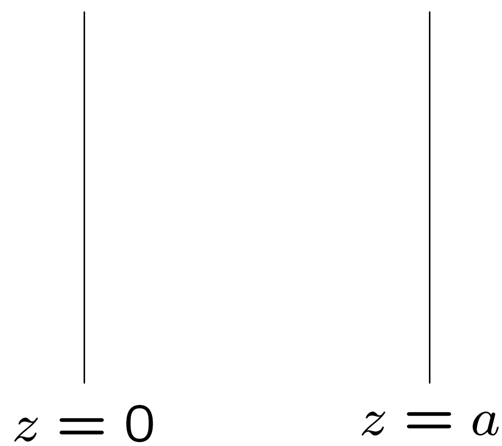
In the vacuum this is divergent and meaningless. What is observable is the *change* in the zero-point energy when matter is introduced. In this way we can calculate the Casimir forces. For a massless scalar field, the canonical energy-momentum tensor is

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}\partial^\lambda\phi\partial_\lambda\phi.$$

The vacuum expectation value may be obtained by taking derivatives of the casual Green's function:

$$G(\mathbf{x},t;\mathbf{x}',t') = \frac{i}{\hbar}\langle \mathsf{T}\phi(\mathbf{x},t)\phi(\mathbf{x}',t')\rangle.$$

Alternatively, we can calculate the stress on the material bodies. Figure 1 shows the original geometry considered by Casimir, where he calculated the quantum fluctuation force between parallel, perfectly conducting plates.*



The force per unit area $\mathcal{F}$ on one of the plates is given in terms of the normal-normal component of the stress tensor,

$$\mathcal{F} = \langle T_{zz} \rangle,$$

*H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948).

For electromagnetic fields, the relevant stress tensor component is

$$T_{zz} = \frac{1}{2}(H_\perp^2 - H_z^2 + E_\perp^2 - E_z^2).$$

We impose classical boundary conditions on the surfaces,

$$H_z = 0, \quad \mathbf{E}_\perp = 0,$$

and the calculation of the vacuum expectation value of the field components reduces to finding the classical TE and TM Green's functions. In general, one further has to subtract off the stress outside the plates, and then the result of a simple calculation, which is sketched below, is

$$f = [T_{zz}(a-) - T_{zz}(a+)] = -\frac{\pi^2}{240a^4}\hbar c,$$
$$= -8.11\,\text{MeV fm}\,a^{-4} = -1.30 \times 10^{-27}\text{N m}^2\,a^{-4},$$

an attractive force.

and will be overwhelmed by electrostatic repulsion between the plates if each plate has an excess electron density n greater than $1/a^2$, from which it is clear that the experiment must be performed at the μm level. For this reason, early measurements were quite inconclusive.* (The cited measurements include insulators as well as conducting surfaces.)

*B. V. Deriagin and I. I. Abrikosova, Zh. Eksp. Teor. Fiz. **30**, 993 (1956); **31**, 3 (1956). [English transl.: Soviet Phys. JETP **3**, 819 (1957); **4**, 2 (1957).]; B. V. Deriagin, I. Abrikosova, and E. M. Lifshitz, Quart. Rev. **10**, 295 (1968); A. Kitchener and A. P. Prosser, Proc. Roy. Soc. (London) A **242**, 403 (1957);M. Y. Sparnaay, Physica **24**, 751 (1958); W. Black, J. G. V. de Jongh, J. T. G. Overbeck, and M. J. Sparnaay, Trans. Faraday Soc. **56**, 1597 (1960); A. van Silfhout, Proc. Kon. Ned. Akad. Wetensch. B **69**, 501 (1966); D. Tabor and R. H. S. Winterton, Nature **219**, 1120 (1968); D. Tabor and R. H. S. Winterton, Proc. Roy. Soc. (London) A **312**, 435 (1969); R. H. S. Winterton, Contemp. Phys. **11**, 559 (1970); J. N. Israelachivili and D. Tabor, Proc. Roy. Soc. (London) A **331**, 19 (1972). For a review of the early experimental work, see Sparnaay's essay in Casimir's commemorative volume, North-Holland, 1989.

Until recently, the most convincing experimental evidence comes from the study of thin helium films;[*] there the corresponding Lifshitz theory[†] has been confirmed over nearly 5 orders of magnitude in the van der Waals potential (nearly two orders of magnitude in distance). In the last few years, the Casimir effect between conductors has been confirmed to the 5% level by Lamoreaux,[‡] and to 1% by Mohideen and Roy.[§] For a review of the experimental situation, see [¶] The Proceedings

[*] E. S. Sabisky and C. H. Anderson, Phys. Rev. A **7**, 790 (1973).

[†] E. M. Lifshitz, Zh. Eksp. Teor. Fiz. **29**, 94 (1956). [English transl.: Soviet Phys. JETP **2**, 73 (1956)]; I. D. Dzyaloshinskii, E. M. Lifshitz, and L. P. Pitaevskii, Usp. Fiz. Nauk **73**, 381 (1961). [English transl.: Soviet Phys. Usp. **4**, 153 (1961).]

[‡] S. K. Lamoreaux, Phys. Rev. Lett. **78**, 6 (1997) **81**, 5475(E); Phys. Rev. A **59**, R3149 (1999).

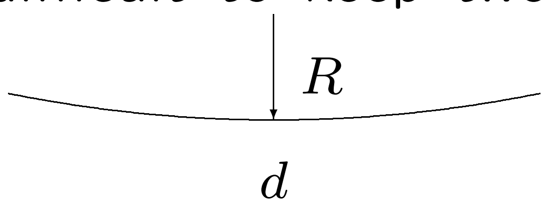
[§] U. Mohideen and A. Roy. Phys. Rev. Lett. **81**, 4549 (1998); A. Roy, C.-Y. Lin, and U. Mohideen, Phys. Rev. **60**, 111101 (1999).

[¶] M. Bordag, U. Mohideen, and V. M. Mostepanenko, Phys. Rep. **353**, 1 (2001).

of QFEXT03 will contain the most recent ex-
perimental updates.

Proximity Force Theorem

Most of the recent experiments measure the force between a flat surface and a spherical surface, both coated with a thin metal surface, such as gold. The reason for this is that it is very difficult to keep two plates exactly



parallel. If $\mathcal{E}(d)$ is the energy of interaction between two plates separated by the distance d, the proximity force theorem says that the force between the plate and the sphere is

$$F = 2\pi R\mathcal{E}(d), \quad R \gg d$$

Thus the Casimir force is

$$F = -\frac{\pi^3}{360}\frac{R}{d^3}$$

Dimensional Dependence
Parallel Plates

Here we wish to concentrate on dimensional dependence. For simplicity we consider a massless scalar field ϕ confined between two parallel plates separated by a distance a. (See Fig. 1.) Assume the field satisfies Dirichlet boundary conditions on the plates, that is

$$\phi(0) = \phi(a) = 0.$$

The Casimir force between the plates results from the zero-point energy per unit (d-dimensional) transverse area

$$\mathcal{E} = \frac{1}{2}\sum \hbar\omega = \frac{1}{2}\sum_{n=1}^{\infty}\int \frac{d^dk}{(2\pi)^d}\sqrt{k^2 + \frac{n^2\pi^2}{a^2}},$$

where we have set $\hbar = c = 1$, and introduced normal modes labeled by the positive integer n and the transverse momentum k.

This may be easily evaluated by introducing a proper-time representation for the square root, and by analytically continuing from negative d we obtain

$$\mathcal{E} = -\frac{1}{2^{d+2}\pi^{d/2+1}}\frac{1}{a^{d+1}}\Gamma\left(1+\frac{d}{2}\right)\zeta(2+d).$$

which reduces to the familiar Casimir result at $d=2$:

$$\mathcal{E} = -\frac{\pi^2}{1440}\frac{1}{a^3}, \quad \mathcal{F}_s = -\frac{\partial}{\partial a}\mathcal{E} = -\frac{\pi^2}{480}\frac{1}{a^4}.$$

This is, as expected, 1/2 of the electromagnetic result.

This less-than-rigorous calculation can be put on a firm footing by a Green's function technique. Define a reduced Green's function by

$$G(x,x') = \int \frac{d^dk}{(2\pi)^d} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} g(z,z'),$$

the (interior) solution of which vanishing at $x=0$, a, being

$$g(z,z') = -\frac{1}{\lambda \sin \lambda a} \sin \lambda z_< \sin \lambda (z_> - a),$$

where $z_>$ ($z_<$) is the greater (lesser) of z and z'. The force per unit area on the surface $z=a$ is obtained by taking the discontinuity of the nnormal-normal component of the stress tensor:

$$\mathcal{F} = \langle T_{zz} \rangle|_{z=z'=a-} - \langle T_{zz} \rangle|_{z=z'=a+}$$
$$= \int \frac{d^dk}{(2\pi)^d} \int \frac{d\omega}{2\pi} \frac{\lambda}{2} (i \cot \lambda a - 1).$$

This is easily evaluated by doing a complex rotation in frequency: $\omega \to i\zeta$:

$$\mathcal{F} = -(d+1)2^{-d-2}\pi^{-d/2-1}\frac{\Gamma\left(1+\frac{d}{2}\right)\zeta(d+2)}{a^{d+2}}.$$

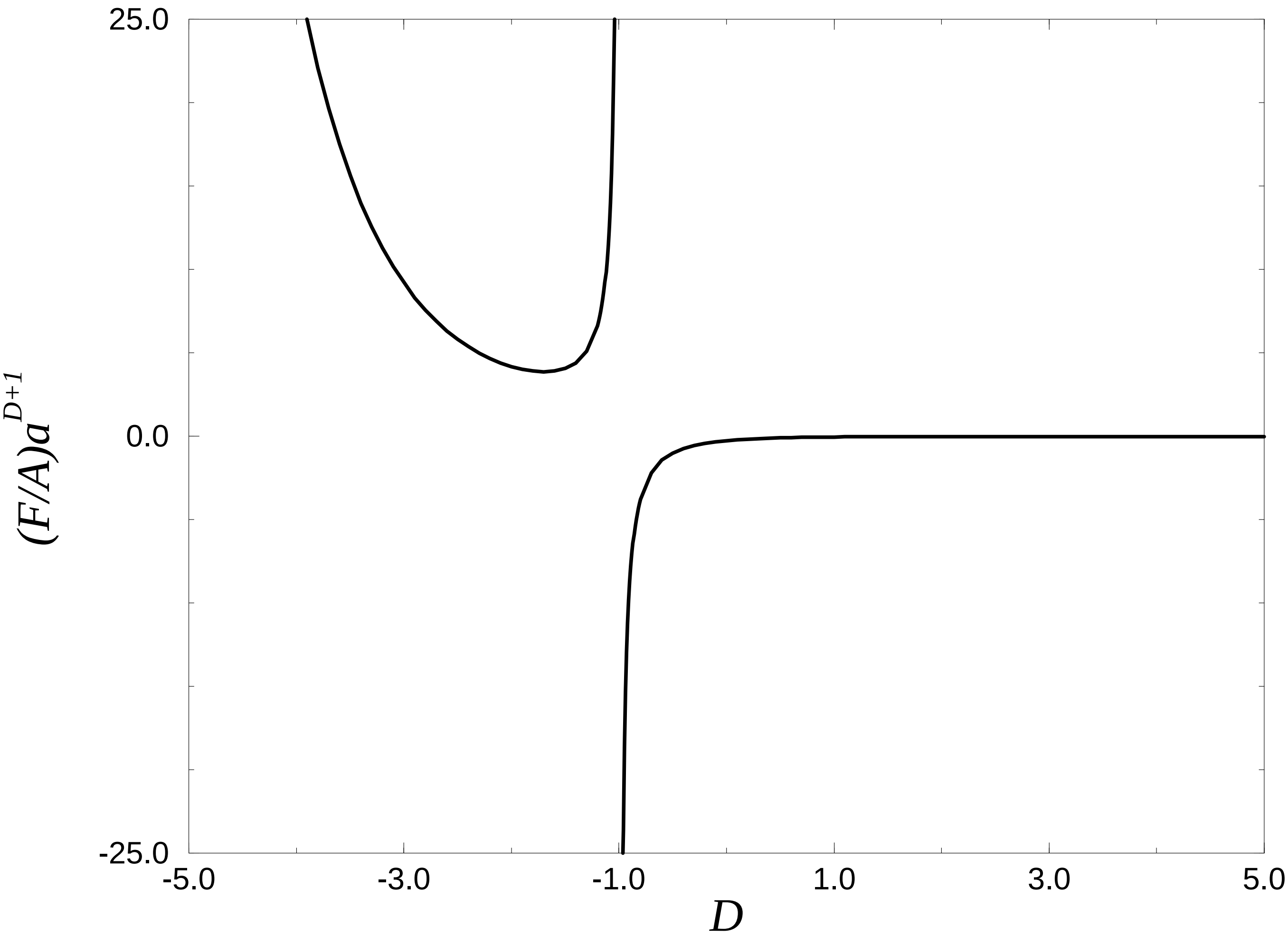
Evidently, this is the negative derivative of the Casimir energy with respect to the separation between the plates:

$$\mathcal{F} = -\frac{\partial \mathcal{E}}{\partial a};$$

this result has now been obtained by a completely well-defined approach. The force per unit area is plotted in Fig. 2, where $a \to 2a$ and $d = D - 1$.

This result was first derived by Ambjørn and Wolfram.*

*J. Ambjørn and S. Wolfram, Ann. Phys. (N. Y.) **147**, 1 (1983).


25.0
0.0
-25.0
-5.0
-3.0
-1.0
1.0
3.0
5.0
$(F/A)a^{D+1}$
D

Fermion fluctuations

The effect of massless fermionic fluctuations, subject to "bag model" boundary conditions,

$$(1 + i\mathbf{n}\cdot\boldsymbol{\gamma})G|_{z=0,a} = 0,$$

where $\mathbf{n}$ is the normal to the surface, was first calculated by Ken Johnson.* The result is 7/4 times the scalar Casimir energy,

$$\mathcal{F}_F = -\frac{7\pi^2}{1920a^4}.$$

*K. A. Johnson, Acta Phys. Pol. **B6**, 865 (1975).

Casimir Effect on a D-dimensional Sphere

Because of the rather mysterious dependence of the sign and magnitude of the Casimir stress on the topology and dimensionality of the bounding geometry, we have carried out calculation of TE and TM modes bounded by a spherical shell in D spatial dimensions. We first consider massless scalar modes satisfying Dirichlet boundary conditions on the surface, which are equivalent to electromagnetic TE modes. Again we calculate the vacuum expectation value of the stress on the surface from the Green's function.

The Green's function $G(\mathbf{x}, t; \mathbf{x}', t')$ satisfies the inhomogeneous Klein-Gordon equation, or

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(\mathbf{x}, t; \mathbf{x}', t') = \delta^{(D)}(\mathbf{x} - \mathbf{x}')\delta(t - t'),$$

where ∇^2 is the Laplacian in D dimensions. We will solve the above Green's function equation by dividing space into two regions, the interior of a sphere of radius a and the exterior of the sphere. On the sphere we will impose Dirichlet boundary conditions

$$G(\mathbf{x}, t; \mathbf{x}', t') \mid_{|\mathbf{x}|=a} = 0.$$

In addition, in the interior we will require that G be finite at the origin $\mathbf{x} = 0$ and in the exterior we will require that G satisfy outgoing-wave boundary conditions at $|\mathbf{x}| = \infty$, that is, for a given frequency, $G \sim e^{ikr}/r$.

The radial Casimir force per unit area $\mathcal{F}$ on the sphere is obtained from the radial-radial component of the vacuum expectation value of the stress-energy tensor:

$$\mathcal{F} = \langle 0 | T^{rr}_{\text{in}} - T^{rr}_{\text{out}} | 0 \rangle \,|_{r=a} \,.$$

To calculate $\mathcal{F}$ we exploit the connection between the vacuum expectation value of the stress-energy tensor $T^{\mu\nu}(\mathbf{x}, t)$ and the Green's function at equal times $G(\mathbf{x}, t; \mathbf{x}', t)$:

$$\mathcal{F} = \frac{1}{2i} \Big[\frac{\partial}{\partial r} \frac{\partial}{\partial r'} G(\mathbf{x}, t; \mathbf{x}', t)_{\text{in}} \\ - \frac{\partial}{\partial r} \frac{\partial}{\partial r'} G(\mathbf{x}, t; \mathbf{x}', t)_{\text{out}} \Big]_{\mathbf{x}=\mathbf{x}',\ |\mathbf{x}|=a} \,.$$

Adding the interior and the exterior contributions, and performing the usual imaginary frequency rotation, we obtain the final expression for the stress*:

$$\mathcal{F} = -\sum_{n=0}^{\infty} \frac{(n-1+\frac{D}{2})\Gamma(n+D-2)}{2^{D-1}\pi^{\frac{D+1}{2}}a^{D+1}n!\,\Gamma\left(\frac{D-1}{2}\right)}$$
$$\times \int_0^\infty dx \left[x\frac{d}{dx}\ln\left(I_{n-1+\frac{D}{2}}(x)K_{n-1+\frac{D}{2}}(x)x^{2-D}\right)\right].$$

It is easy to check that this reduces to the known case at $D = 1$, for there the series truncates—only $n = 0$ and 1 contribute, and we easily find

$$\mathcal{F}_{D=1} = \frac{F}{2} = -\frac{\pi}{96a^2},$$

which agrees with parallel plate result for $d = 0$ and $a \to 2a$.

*C. M. Bender and K. A. Milton, Phys. Rev. D **50**, 6547 (1994)

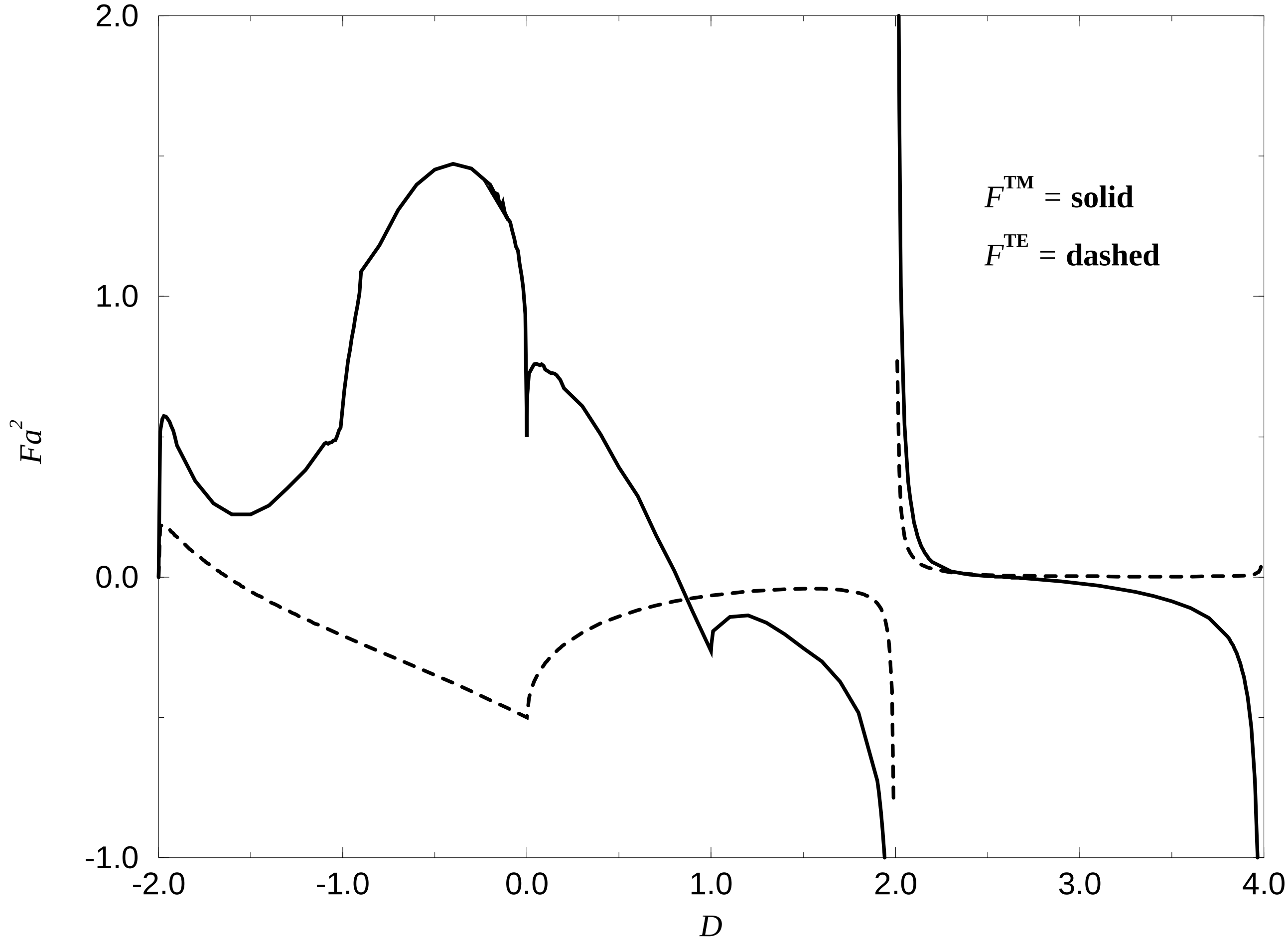
We proceed as follows:

- Analytically continue to $D < 0$, where the sum converges, although the integrals become complex.

- Add and subtract the leading asymptotic behavior of the integrals.

- Continue back to $D > 0$, where everything is now finite.

The results of numerical evaluations are as shown in Fig. 3.



2.0
1.0
0.0
-1.0
Fa^2
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0
D
F^{TM} = **solid**
F^{TE} = **dashed**

Note the following features for the scalar modes:

- Poles occur at $D = 2n$, $n = 1, 2, 3, \ldots$.

- Branch points occur at $D = -2n$, $n = 0, 1, 2, 3, \ldots$, and the stress is complex for $D < 0$.

- The stress vanishes at negative even integers, $F(-2n) = 0$, $n = 1, 2, 3, \ldots$, but is nonzero at $D = 0$: $F(0) = -1/2a^2$.

- The case of greatest physical interest, $D = 3$, has a finite stress, but one which is much smaller than the corresponding electrodynamic one: $F(3) = +0.0028168/a^2$. (This result was confirmed in Ref. *

*S. Leseduarte and A. Romeo, Europhys. Lett. **34**, 79 (1996); Ann. Phys. (N.Y.) **250**, 448 (1996).

The TM modes are modes which satisfy mixed boundary conditions on the surface,

$$\frac{\partial}{\partial r} r^{D-2} G(\mathbf{x}, t; \mathbf{x}', t')|_{|\mathbf{x}|=r=a} = 0,$$

The results are qualitatively similar, and are also shown in Fig. 3. In particular, removing the $n = 0$ contribution from the sum of the TE and TM contributions, we recover the repulsive Boyer result,*

$$E_{\text{sphere}} = \frac{0.092353}{2a}.$$

For the 3-sphere, the fermionic Casimir energy is considerably smaller†

$$E_F = \frac{0.0204}{a}.$$

*T. H. Boyer, Phys. Rev. **174**, 1764 (1968).

†K. Johnson, unpublished; K. A. Milton, Ann. Phys. (N.Y.) **150**, 432 (1983).

Cylinders

A similar calculation of the electromagnetic Casimir effect of a perfectly conducting infinite right circular cylinder was performed by DeRaad and me twenty years ago. The calculation is rather more involved, and the regularization of the divergences more subtle. The result is* for the Casimir energy per unit length, or the force per unit area,

$$\mathcal{E} = \pi a^2 \mathcal{F} = -0.01356/a^2.$$

Unlike the three-dimensional sphere, the cylinder experiences an attractive Casimir stress. Two recent calculations have confirmed this result using zeta-function techniques.†

*L. L. DeRaad, Jr. and K. A. Milton, Ann. Phys. (N.Y.) **136**, 229 (1981).

†P. Gosdzinsky and A. Romeo, Phys. Lett. B **441** (1998); K. A. Milton, A. V. Nesterenko, and V. V. Nesterenko, Phys. Rev. D **59**, 105009 (1999).

Force between dielectric slabs

Over 40 years ago, Lifshitz and collaborators worked out the corresponding forces between dielectric slabs. Imagine we have a permittivity which depends on z as follows:

$$\epsilon(z) = \begin{cases} \epsilon_1, & z < 0, \\ \epsilon_3, & 0 < z < a, \\ \epsilon_2, & a < z. \end{cases} .$$

Then the Lifshitz force between the bodies at zero temperature is given by ($\kappa^2 = k^2 + \epsilon\zeta^2$)

$$\mathcal{F}^{T=0}_{\text{Casimir}} = -\frac{1}{8\pi^2}\int_0^\infty d\zeta \int_0^\infty dk^2\, 2\kappa_3 \left(d^{-1} + d'^{-1}\right).$$

Here the denominators are given by, for the electric Green's function,

$$d = \frac{\kappa_3 + \kappa_1}{\kappa_3 - \kappa_1}\frac{\kappa_3 + \kappa_2}{\kappa_3 - \kappa_2}e^{2\kappa_3 a} - 1.$$

The magnetic Green's function g^H has the same form but with the replacement

$$\kappa \to \kappa/\epsilon \equiv \kappa',$$

in the reflection coefficients multiplying the exponentials; the corresponding denominator is denoted by d'. From this, we can obtain the finite temperature expression immediately by the substitution

$$\zeta^2 \to \zeta_n^2 = 4\pi^2 n^2/\beta^2,$$

$$\int_0^\infty \frac{d\zeta}{2\pi} \to \frac{1}{\beta}\sum_{n=0}^{\infty}{}',$$

the prime being a reminder to count the $n = 0$ term with half weight. (More about this later!)

Relation to van der Waals force

If the central medium is tenuous, $\epsilon - 1 \ll 1$, and is surrounded by vacuum, for large distances $a \gg \lambda_c$, we can expand the above general formula and obtain a dispersion-free result:

$$\mathcal{F} \approx -\frac{23(\epsilon - 1)^2}{640\pi^2 a^4}.$$

For this regime, this should be derivable from the sum of van der Waals forces, obtained from a potential of the form

$$V = -\frac{B}{r^\gamma}.$$

We do this by computing the energy (N = density of molecules)

$$E = -\frac{1}{2} B N^2 \int_0^a dz \int_0^a dz' \times \int \frac{(d\mathbf{r}_\perp)(d\mathbf{r}'_\perp)}{[(\mathbf{r}_\perp - \mathbf{r}'_\perp)^2 + (z - z')^2]^{\gamma/2}}. \quad (1)$$

If we disregard the infinite self-interaction terms (analogous to dropping the volume energy terms in the Casimir calculation), we get

$$\mathcal{F} = -\frac{\partial}{\partial a}\frac{E}{A} = -\frac{2\pi B N^2}{(2-\gamma)(3-\gamma)}\frac{1}{a^{\gamma-3}}.$$

Then, we set $\gamma = 7$ and in terms of the polarizability,

$$\alpha = \frac{\epsilon - 1}{4\pi N},$$

we recover the Casimir-Polder retarded dispersion potential*,

$$V = -\frac{23}{4\pi}\frac{\alpha^2}{r^7},$$

whereas for short distances we recover the London potential,†

$$V = -\frac{3}{\pi}\frac{1}{r^6}\int_0^\infty d\zeta\,\alpha(\zeta)^2.$$

*H. B. G. Casimir and D. Polder, Phys. Rev. **73**, 360 (1948).

†F. London, Z. Physik **63**, 245 (1930).

Given the divergences of the above calculation, and the essentially one-dimensional restriction, it is of interest to consider a tenuous dielectric sphere. The theory of the Casimir energy for a dielectric ball was first worked out by me 20 years ago.* The general expression is of course quite complicated

$$E = -\frac{1}{4\pi a}\int_{-\infty}^{\infty} dy\, e^{iy\delta} \sum_{l=1}^{\infty} (2l+1) x \frac{d}{dx} \ln S_l,$$

where

$$S_l = [s_l(x')e_l'(x) - s_l'(x')e_l(x)]^2 \\ -\xi^2 [s_l(x')e_l'(x) + s_l'(x')e_l(x)]^2, \qquad (2)$$

where the s_l , e_l are spherical Bessel functions of imaginary argument, the quantity ξ is

$$\xi = \frac{\sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}} - 1}{\sqrt{\frac{\epsilon'\mu}{\epsilon\mu'}} + 1},$$

*K. A. Milton, Ann. Phys. (N.Y.) **127**, 49 (1980).

where ϵ', μ' represent the permittivity and permeability in the interior, the time-splitting parameter is denoted by δ, and

$$x = |y|\sqrt{\epsilon\mu}, \quad x' = |y|\sqrt{\epsilon'\mu'}.$$

It is easy to check that this result reduces to that for a perfectly conducting spherical shell if we set the speed of light inside and out the same, $\sqrt{\epsilon\mu} = \sqrt{\epsilon'\mu'}$, as well as set $\xi = 1$. However, if the speed of light is different in the two regions, the result is no longer finite, but quartically divergent, and indeed the Schwinger result* follows for that leading divergent term.

*J. Schwinger, Proc. Natl. Acad. Sci. USA **90**, 958, 2105, 4505, 7285 (1993); **91**, 6473 (1994).

Unlike the situation for a shell of negligible thickness, this expression is not finite. However, there are several methods of isolating the divergences, at least if the ball is tenuous, ($\epsilon - 1 \ll 1$), and the finite leading result is*

$$E_{\text{Cas}} = \frac{23}{1536\pi a}(\epsilon - 1)^2.$$

What is most remarkable about this result is that it coincides with the van der Waals energy calculated two years earlier for this nontrivial geometry. That is, starting from the Casimir-Polder potential we summed the pairwise potentials between molecules making up the media.

*I. Brevik, V. N. Marachevsky, and K. A. Milton, Phys. Rev. Lett. **82**, 3948 (1999); G. Barton, J. Phys. A **32**, 525 (1999); M. Bordag, K. Kirsten, and D. Vassilevich, Phys. Rev. D **59**, 085011 (1999); J. S. Høye and I. Brevik, quant-ph/9903086; J. Statistical Phys.

A sensible way to regulate this calculation is dimensional continuation, similar to that described above. That is, we evaluate the integral

$$E_{\text{vdW}} = -\frac{23}{8\pi}\alpha^2 N^2 \int d^D r\, d^D r' (r^2 + r'^2 - 2rr' \cos\theta)^{-\gamma/2},$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{r}'$, by first regarding $D > \gamma$ so the integral exists. The integral may be done exactly in terms of gamma functions, which when continued to $D = 3$, $\gamma = 7$ yields the Casimir result.*

$$E_{\text{vdW}} = E_{\text{Cas}}$$

*K. A. Milton and Y. J. Ng, Phys. Rev. E **57**, 5504 (1998).

Thus there can hardly be any doubt that the Casimir effect, in the tenuous limit, coincides with the van der Waals attraction between molecules. This seems to go some way toward providing understanding of this zero-point fluctuation phenomenon. But the subject is not closed. Five years ago Romeo and I demonstrated that for a dilute cylinder the van der Waals energy is zero.* Presumably the same hold for the Casimir energy, but a convincing demonstration of that is not yet at hand. Remarkably, for a dilute cylinder with constant speed of light inside and out, $\epsilon\mu =$ constant, it has been demonstrated that the Casimir energy vanishes, but that would seem to be a completely different case.

*A. Romeo, private communication; K. A. Milton, A. V. Nesterenko, and V. V. Nesterenko, Phys. Rev. D **59**, 105009 (1999)

Temperature Dependence of Casimir Effect Between Real Metals

The Casimir surface force density $\mathcal{F}^T$ between two identical dielectric plates separated by a distance a can be written as the Lifshitz formula

$$\mathcal{F}^T = -\frac{1}{\pi\beta}\sum_{m=0}^{\infty}{}' \int_{\zeta_m}^{\infty} q^2 dq \Big[\frac{A_m e^{-2qa}}{1 - A_m e^{-2qa}} + \frac{B_m e^{-2qa}}{1 - B_m e^{-2qa}}\Big].$$

The relation between q and the transverse wave vector $\mathbf{k}_\perp$ is $q^2 = k_\perp^2 + \zeta_m^2$, where $\zeta_m = 2\pi m/\beta$. Furthermore, the squared reflection coefficients are

$$\text{TM:} \quad A_m = \left(\frac{\varepsilon p - s}{\varepsilon p + s}\right)^2, \quad \text{TE:} \quad B_m = \left(\frac{s-p}{s+p}\right)^2,$$

$$s^2 = \varepsilon - 1 + p^2, \quad p = \frac{q}{\zeta_m},$$

with $\varepsilon(i\zeta_m)$ being the permittivity.

For an ideal metal, $\varepsilon \to \infty$, we set all the reflection coefficients equal to unity: $A_m = B_m = 1$. Then for low temperature the force/area between the plates is

$$\mathcal{F}^T = -\frac{\pi^2}{240a^4}\left[1 + \frac{1}{3}\left(\frac{2a}{\beta}\right)^4\right], \quad (aT \ll 1)$$

The free energy and the internal energy can both be obtained directly:

$$F = -\frac{\pi^2}{720a^3}\left(1 + 45\left(\frac{2a}{\beta}\right)^3\frac{\zeta(3)}{\pi^3} - \left(\frac{2a}{\beta}\right)^4\right).$$

$$U = -\frac{\pi^2}{720a^3}\left[1 - 90\left(\frac{2a}{\beta}\right)^3\frac{\zeta(3)}{\pi^3} + 3\left(\frac{2a}{\beta}\right)^4\right].$$

The entropy thus becomes (recall that $B_0 = 1$ is assumed)

$$S = \frac{U - F}{T} \sim \frac{3\zeta(3)}{2\pi}T^2 - \frac{4\pi^2 a}{45}T^3, \quad aT \ll 1.$$

However, real metals are not described by this ideal limit $\epsilon \to 0$. If we attempt to do better by setting $B_0 = 0$ (no TE zero mode) we obtain a linear temperature correction

$$\mathcal{F}^T = \frac{\beta}{2\pi}\int_0^\infty d\zeta\, f(\zeta) + \frac{\zeta(3)}{8\pi\beta a^3} - \frac{\pi^2}{45}\left(\frac{a}{\beta}\right)^4$$
$$= -\frac{\pi^2}{240a^4}\left[1 + \frac{16}{3}\left(\frac{a}{\beta}\right)^4\right] + \frac{\zeta(3)}{8\pi a^3}T.$$

There are apparently two serious problems with this result:

It would seem to be ruled out by experiment. The ratio of the linear term to the $T = 0$ term is

$$\Delta = \frac{30\zeta(3)}{\pi^3} aT = 1.16 aT,$$

or putting in the numbers (300 K = $(38.7)^{-1}$ eV, $\hbar c = 197$ MeV fm)

$$\Delta = 0.15 \left(\frac{T}{300 \text{ K}}\right) \left(\frac{a}{1 \mu\text{m}}\right),$$

or as Klimchitskaya observed, there is a 15% effect at room temperature at a separation of one micron. One would have expected this to have been been seen by Lamoreaux; his experiment was reported to be in agreement with the conventional theoretical prediction at the level of 5%.

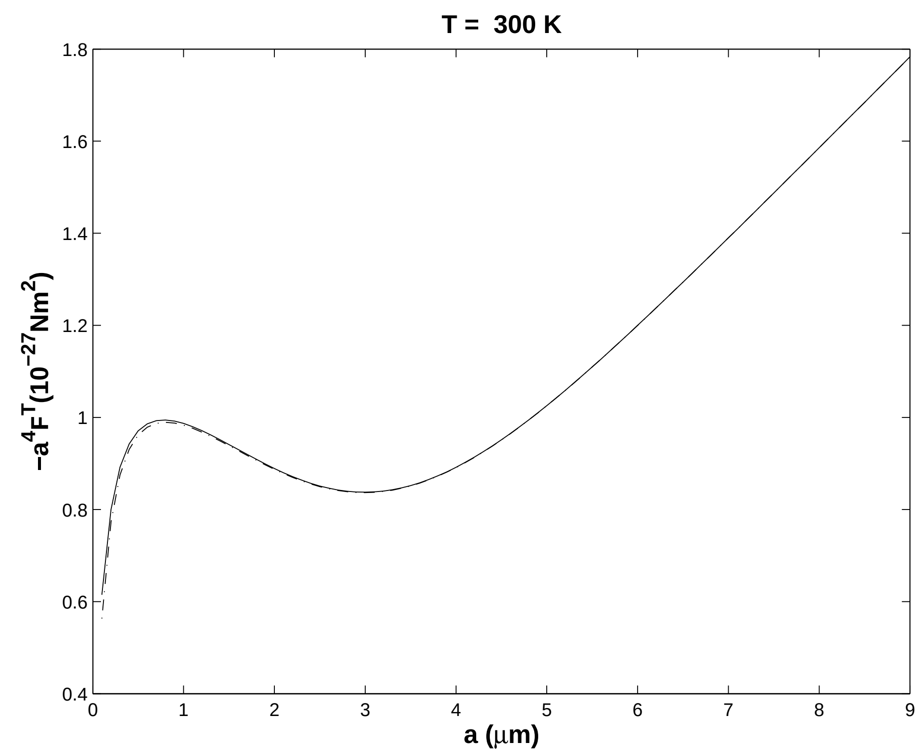
Another apparently serious problem is the apparent thermodynamic inconsistency. A linear term in the force implies a linear term in the free energy (per unit area),

$$F = F_0 + \frac{\zeta(3)}{16\pi a^2} T, \quad aT \ll 1,$$

which implies a nonzero contribution to the entropy/area at zero temperature:

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = -\frac{\zeta(3)}{16\pi a^2}.$$

Taken at face value, this statement appears to be incorrect. We will discuss this problem more closely below, and will find that although a linear temperature dependence will occur at room temperature (see the following figure), the entropy will go to zero as the temperature goes to zero. The point is that the free energy F for a finite ε always will have a zero slope at $T = 0$, thus ensuring that $S = 0$ at $T = 0$. The apparent contradiction is due to the fact that the curvature of $F(T)$ near $T = 0$ becomes infinite when $\varepsilon \to \infty$.



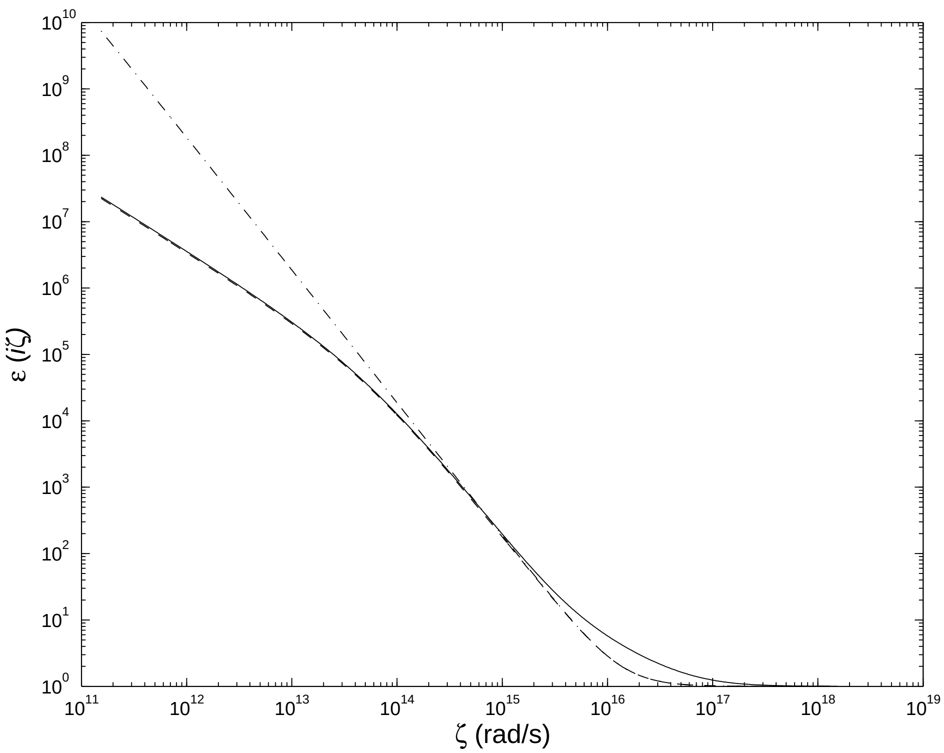
Surface force density for gold, multiplied with a^4, versus a when $T = 300$ K. Input data for $\varepsilon(i\zeta)$ are taken from Lambrecht and Reynaud.

A detailed analysis can be based on describing the permittivity of a metal by the Drude model, valid for low frequencies,

$$\varepsilon(i\zeta) = 1 + \frac{\omega_p^2}{\zeta(\zeta + \nu)},$$

where ω_p is the plasma frequency and ν the relaxation frequency, which is quite accurate at low frequencies: See Figure. The TE reflection coefficient is then

$$B(\zeta, p) = \left(\sqrt{1 + \frac{\nu\zeta}{\omega_p^2} p^2} - \sqrt{\frac{\nu\zeta}{\omega_p^2}} p\right)^4, \quad \zeta \ll 1.$$



Full line: Permittivity $\varepsilon(i\zeta)$ *as function of imaginary frequency* ζ *for gold. The curve is calculated on the basis of experimental data. Courtesy of Astrid Lambrecht and Serge Reynaud. Broken lines:* $\varepsilon(i\zeta)$ *versus* ζ *with* T *as parameter, based upon the temperature dependent Drude model. The upper curve is for* $T = 10$ *K; the lower is for* $T = 300$ *K, which for energies below 1 eV* (1.5×10^{15} *rad/s) nicely fits the experimental data. Both curves are below the experimental one for* $\zeta > 2 \times 10^{15}$ *rad/s.*

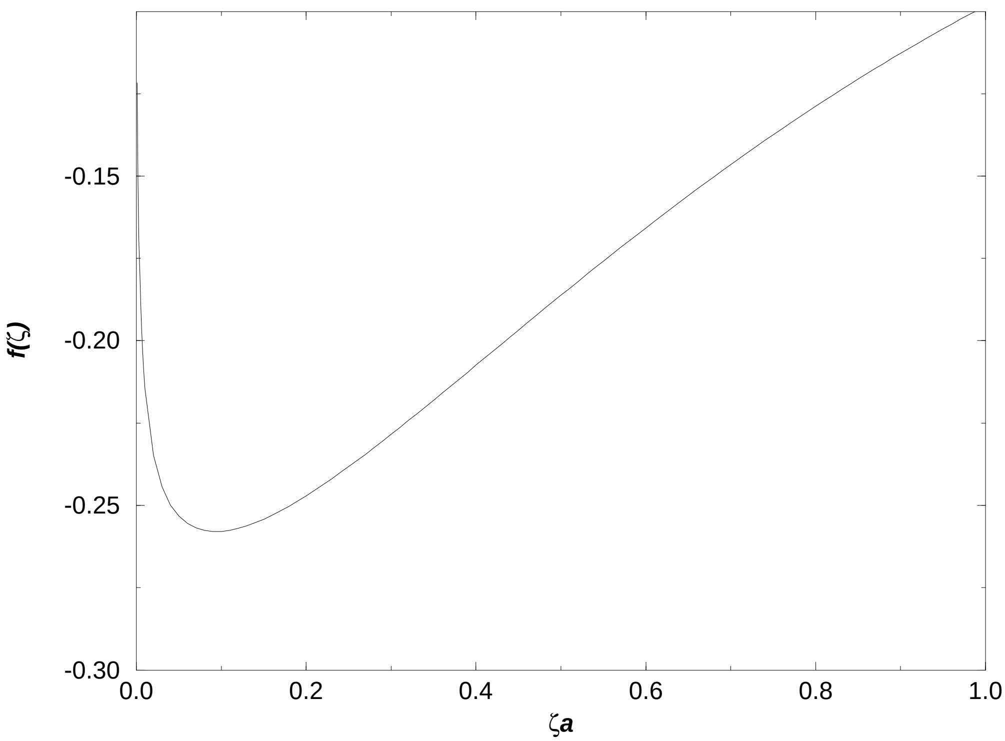
Analytic and numerical calculations show the free energy has a quadratic temperature dependence,

$$F(T) = F_0 + T^2 \frac{\omega_p^2}{6\nu}(2\ln 2 - 1) = F_0 + T^2(19 \text{ eV}),$$

putting in the numbers for gold, rather than the naive extrapolation

$$F = F_0 + T\frac{\zeta(3)}{16\pi a^2}$$

See Figure.



The behavior of the free energy for small frequency, in the Drude model, with parameters suitable for gold. Here

$$F^{\mathrm{TM}} = \frac{T}{2\pi} \sum_{m=0}^{\infty}{}' f(\zeta_m).$$

Casimir Effect and Renormalized QFT

Recently, Graham et al.* have questioned the above results for ideal boundaries. They have developed an approach in which idealized boundary conditions are replaced with interactions with an external (nondynamical) field. Potentially divergent terms are subtracted and replaced by perturbatively calculable Feynman diagrams. After renormalization of these diagrams, the limiting case when the external field becomes a delta function is taken. In this way the results for $D = 1$ are reproduced; but the authors find those finite results rather unsatisfactory, so they discuss how their limiting procedure gives rise to a different energy, corresponding, however, to the conventional force.

*N. Graham, R. L. Jaffe, M. Quandt, and H. Weigel, Phys. Rev. Lett. **87**, 131601 (2001); N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, M. Scandurra, and H. Weigel, Int. J. Mod. Phys. A **17**, 846 (2002); hep-th/0207205; Nucl. Phys. B **645**, 49 (2002); N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, O. Schröder, and H. Weigel, hep-th/0309130.

Then they turned to $D = 2$ and find that it is divergent; the implication is that this is a general feature, so that all calculations of Casimir self-stress are called into question.

However, as we remarked above, $D = 2$ is a singular point. I have,* and very recently also, the MIT group have reexamined the $D = 3$ calculation for a sphere. Instead of imposing boundary conditions, we consider an interaction with a background field σ,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2 + \sigma(r)\phi^2),$$

where at the end we could take the external field to be a delta function

$$\sigma(r) = \frac{g}{a}\delta(r - a),$$

which would correspond to the imposition of Dirichlet boundary conditions.

*K. A. Milton, Phys. Rev. D **68**, 065020 (2003)

The key calculation is that of the energy computed to second order:

$$E = \frac{i}{2T} \mathrm{Tr}\, \sigma G_0 \sigma G_0$$
$$= -\frac{1}{2^{2D+2}\pi^{D+1/2}} \frac{\Gamma\left(\frac{3-D}{2}\right)}{\Gamma\left(\frac{D}{2}\right)} \int_0^\infty dk\, k^{D-1} \tilde{\sigma}(k)^2$$
$$\times \int_0^1 du\, [m^2 + u(1-u)k^2]^{D/2-3/2},$$

which exactly coincides with the $D = 2$ and $D = 3$ results of Graham et al.

However, they claim that in the sharp limit this is obviously divergent for $D \geq 2$, which divergence cannot be removed by any sort of renormalization.

I claim, on the contrary, that the divergence does not contribute to the stress on the sphere, and, for example, by dimensional continuation, the energy for a massless scalar is

$$E=-\frac{g^2}{a}\frac{\Gamma\left(\frac{D-1}{2}\right)\Gamma(D-3/2)\Gamma(1-D/2)}{2^{1+2D}\pi\Gamma\left(\frac{D}{2}\right)^2},$$

which we take to be the appropriate analytic continuation for all D. This exhibits poles at $D=2,4,6,\ldots$, in congruence with the known divergence structure of the Casimir effect. There are also poles occurring at $D=1,-1,-3,\ldots$, and at $D=3/2,1/2,-1/2,\ldots$. These latter two sequences of divergent dimensions correspond to infrared divergences that have no counterpart in the Casimir calculations, unlike the ultraviolet, even-integer poles. For space dimension between 2 and 4 the Casimir energy is completely finite, in concert with this diagnostic. The divergence at $D=2$, even putting aside the question of mass, is seen not to be generic.

Alternatively, one can calculate in three-dimensions directly,

$$E = \frac{1}{8\pi^2}\int_0^\infty dx\, x\,\sigma(x)\int_0^\infty dy\, y\,\sigma(y) \times\frac{d}{d\alpha}\int_0^\infty dk\, k^{2\alpha}\sin kx \sin ky\big|_{\alpha=0}.$$

The k integral is now obtained from ($-1 < \alpha < 0$)

$$\int_0^\infty dx\, x^\alpha \cos\beta x = \frac{\Gamma(\alpha+1)\cos(\alpha+1)\frac{\pi}{2}}{\beta^{\alpha+1}},$$

$$E = \frac{1}{16\pi}\int_0^\infty dx\, x\sigma(x)\int_0^\infty dy\, y\sigma(y)\left(\frac{1}{x+y}-\frac{1}{|x-y|}\right) \to \frac{g^2}{32\pi a},$$

where we have omitted another infinite term that is independent of a. The result is exactly the $D = 3$ value found above! The justification for omitting (infinite) constant terms in the energy is that they are unobservable, not corresponding to a self-stress on the sphere.

Conclusions

This history of the Casimir effect, or of macroscopic manifestations of zero-point energy, is given an iconoclastic reading in my recent monograph*, to be updated in a review article which I should complete by the beginning of 2003. It is a sign of the vitality of the subject that there are heated discussions and divergences of opinion. In the coming months we will see great progress:

• The temperature controversies, if not settled by a convergence in theoretical opinion, will be settled by experiment.

• The problems of infinities are under a variety of attacks. It is likely, in my view, that finite, and ultimately observable self-forces will remain. Energies are ambiguous; and perturbative renormalization is not the last word when dealing with complex systems.

*K.A. Milton, The Casimir Effect, World Scientific, 2001