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Analytic perturbation theory and inclusive τ decay

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Abstract

We apply analytic perturbation theory to the inclusive decay of a τ lepton into hadrons. It is shown that the resulting analyticity of the coupling constant strongly influences the value of the QCD Λ-parameter extracted from the experimental data on τ decay.

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I. INTRODUCTION

The inclusive character of the decay of a τ lepton into hadrons and the fact that the nonperturbative QCD contributions to this process are very small [1] make it possible, in principle, to describe this process on the basis of the standard methods of quantum field theory without any model assumptions. Measurement of the ratio of hadronic to leptonic τ decay widths, i.e., the quantity $R_\tau = \Gamma(\tau \to \text{hadrons} + \nu)/\Gamma(\tau \to l\nu\bar{\nu})$, allows one to extract, with a high degree of accuracy, the value of the strong coupling constant $\alpha_{\rm QCD}$ at the τ mass, $Q = M_\tau \simeq 1.78$ GeV. Comparing this value with the values of $\alpha_{\rm QCD}$ found at higher energy, for example, $\alpha_{\rm QCD}(Q^2 = M_Z^2)$, is an important test of the applicability of QCD perturbation theory over a wide range of energies and requires a careful check. At present, both experimental and theoretical investigations of the decay of a τ lepton are continuing intensively (see the recent reviews in [2]).

One usually employs analytic properties of the hadronic correlation function in order to rewrite the original expression for the τ hadronic rate, which involves integration over a nonperturbative region of small momenta, in the form of a contour integral over a circle of sufficiently large radius $Q^2 = M_\tau^2$ to apply perturbation theory (PT) [1,2]. However, the perturbative approximation, which introduces a ghost pole, violates the analytic properties required to use Cauchy's theorem in this manner.

In this paper, we will apply analytic perturbation theory (APT) [3,4] in which it is possible to maintain the correct analytic behavior. Within this approach the correct analytic properties of the running coupling are provided by nonperturbative contributions which emerge automatically from dispersion relations.[1] The analytic running coupling constant obtained in such a way turns out to be remarkably stable for the whole interval of momentum, and has a universal infrared limit at $Q^2 = 0$, independent of the value of the QCD scale parameter Λ.

For our purpose, it is important that, within this approach, it is possible to give a self-consistent definition of the running coupling constant in the Minkowskian (timelike) region [9]. This fact allows us to obtain two equivalent representations for the QCD correction to τ decay, involving the timelike and the spacelike definitions of the running coupling constants, respectively. For our numerical estimations we will use APT at 2-loop order.

II. WHY IS ANALYTICITY IMPORTANT?

The initial theoretical expression for R_τ in the case of massless quarks contains an integral over timelike momentum s [1]

$$R_\tau = \frac{2}{\pi} \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + 2\frac{s}{M_\tau^2}\right) \mathrm{Im}\,\Pi(s)\,, \tag{1}$$

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[1]Some other aspects of the dispersive approach have been discussed in [5–7] and in the recent papers [8].

where the range of integration extends down to small s and cannot be calculated in the framework of the standard perturbation theory. The method of calculation of R_τ based on exploiting certain analytic properties of the hadronic correlators $\Pi(s)$ allows one to rewrite the expression (1) by using the Cauchy theorem in the form of a contour integral in the complex s-plane with the contour running counterclockwise around a circle centered on the origin of radius M_τ^2:

$$R_\tau \;=\; \frac{1}{2\pi \mathrm{i}} \oint_{|s|=M_\tau^2} \frac{ds}{s} \left(1 - \frac{s}{M_\tau^2}\right)^3 \left(1 + \frac{s}{M_\tau^2}\right) D(s)\,, \tag{2}$$

where[2]

$$D(q^2) \;=\; -q^2 \,\frac{d\,\Pi(q^2)}{d\,q^2}\;.$$

In the representation (2) the contour has a sufficiently large radius, and it is possible, in principle, to calculate R_τ perturbatively. However, the transition to the contour representation requires certain analytic properties of the correlator. Namely, the correlator $\Pi(s)$ is an analytic function in the complex s-plane with a cut along the positive part of the real axis. The parametrization of Π by the perturbative running coupling constant violates these analytic properties [10]. It is clear that the difference in the regions of integration in the initial expression (1) for R_τ and the expression (2) obtained after applying the Cauchy theorem makes it necessary to parametrize Π in Eq. (1) and D in Eq. (2) with different coupling constants. Indeed, a renormalization-group analysis gives a running coupling constant determined in the spacelike (Euclidean) region, while the initial expression (1) contains an integration over timelike momentum and therefore to calculate Eq. (2) requires some procedure of analytic continuation from spacelike to timelike momentum. To this end, we will determine the effective coupling constant in the spacelike region (t-channel), $\bar{a}^{\rm eff}$, and in the timelike region, $\bar{a}_s^{\rm eff}$, (using the definition that $a = \alpha_{\rm QCD}/4\pi$) as (where the overbar signifies that the coupling constant has the correct analytic properties)

$$D(q^2) \sim 1 \,+\, d_1\, a(q^2) \,+\, d_2\, a^2(q^2) \,+\cdots \,=\, 1 \,+\, d_1\, \bar{a}^{\rm eff}(q^2)\,, \tag{3}$$

$$\mathrm{Im}\,\Pi(s) \sim 1 \,+\, r_1\, a_s(s) \,+\, r_2\, a_s^2(s) \,+\cdots \,=\, 1 \,+\, r_1\, \bar{a}_s^{\rm eff}(s)\,, \tag{4}$$

where the first perturbative coefficients d_1 and r_1 are equal to each other: $d_1 = r_1 = 4$.

The dispersion relations for the D-function yield the following relation between the effective coupling constants:

$$\bar{a}^{\rm eff}(q^2) = -\,q^2 \int_0^\infty \frac{ds}{(s-q^2)^2}\, \bar{a}_s^{\rm eff}(s)\,, \tag{5}$$

$$\bar{a}_s^{\rm eff}(s) = -\,\frac{1}{2\pi \mathrm{i}} \int_{s-\mathrm{i}\,\epsilon}^{s+\mathrm{i}\,\epsilon} \frac{dz}{z}\, \bar{a}^{\rm eff}(z)\,. \tag{6}$$

Now, separating the QCD contribution Δ_τ in R_τ,

$$R_\tau = 3\,\left(|V_{ud}|^2 + |V_{us}|^2\right)\, S_{\rm EW}\,\left(1 + \Delta_\tau\right)\,, \tag{7}$$

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[2]Here, we will use the standard definition $q^2 = -Q^2$, so that in the Euclidean region $Q^2 > 0$.

where V_{ud} and V_{us} are the CKM matrix elements, and $S_{\rm EW}$ is the electroweak factor (see [1]), we obtain from Eqs. (1) and (2) the two equivalent representations,

$$\Delta_\tau \;=\; 2\,r_1 \int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left(1+2\frac{s}{M_\tau^2}\right)\bar{a}_s^{\rm eff}(s)\,, \tag{8}$$

and

$$\Delta_\tau \;=\; \frac{d_1}{2\pi {\rm i}} \oint_{|z|=M_\tau^2} \frac{dz}{z}\left(1-\frac{z}{M_\tau^2}\right)^3\left(1+\frac{z}{M_\tau^2}\right)\bar{a}^{\rm eff}(z)\,. \tag{9}$$

It should be noted that the equivalence of these formulae holds only in the case of the above-mentioned analytic properties of the correlator $\Pi(s)$ and the D-function and that these analytic properties are broken in the standard perturbation theory. The QCD contribution represented in the form (2) is the expression which one usually uses for theoretical analysis. In principle, the expression (9) can be calculated on the basis of perturbation theory, but then it is impossible to return from Eq. (9) to Eq. (8) which corresponds to the initial Eq. (1), and nothing can be said about the error associated with switching from Eq. (8) to Eq. (9). Therefore, it is impossible to give in the framework of the standard perturbation theory a self-consistent description of the inclusive decay of a τ lepton into hadrons.

III. THE APT METHOD FOR τ-DECAY

Now, we apply analytic perturbation theory (APT), which was recently proposed in [3,4,9], to describe the inclusive decay of a τ lepton into hadrons and to give a quantitative estimate of the effect associated with Q^2-analyticity of the running coupling constant. The method of APT allows one to avoid the above-mentioned difficulties and evaluate both the initial integral over the physical region and the contour representation which are equal due to the Cauchy theorem. In the framework of APT, Π and the D-function can be parametrized by the running coupling constant with the correct analytic properties as in Eqs. (3) and (4), where the running coupling constants in the space- and timelike regions can be expressed in the terms of the spectral density $\varrho(\sigma)$ as [9]

$$\bar{a}(z) = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma - z}\,\varrho(\sigma)\,, \tag{10}$$

$$\bar{a}_s(s) = \frac{1}{\pi}\int_s^\infty \frac{d\sigma}{\sigma}\,\varrho(\sigma)\,. \tag{11}$$

In the leading order, the expression for $\bar{a}(z)$ has the form

$$\bar{a}^{(1)}(z) \;=\; \frac{1}{\beta_0}\left[\frac{1}{\ln(-z/\Lambda^2)} + \frac{1}{1+z/\Lambda^2}\right], \tag{12}$$

where $\beta_0 = 11 - 2n_f/3$ is the first coefficient of the β function. The first term in (12) determines the asymptotic behavior at large momenta and has the same form as in perturbation theory. The second term, which appears automatically, reproduces the correct analytic properties, and the ghost pole at $z = -\Lambda^2$ does not arise. We underscore once again that APT makes it possible to implement the correct transition from expression (1) to expression (2). These expressions simply coincide, as they should, while the application of perturbation theory with the standard renormalization-group refinement runs into serious difficulties.

The fundamental quantity in the APT approach is the spectral density $\varrho(\sigma)$ by which one can parametrize both the running coupling constants in the spacelike and in the timelike regions. We now find a formula which expresses the strong interaction contribution to R_τ via the spectral density function. To this end, introduce an effective spectral density $\rho^{\rm eff}(\sigma)$ that corresponds to the effective coupling constant [see Eq. (3)] and write down the effective coupling constants in the Euclidean and physical regions as follows,

$$\bar{a}^{\rm eff}(z) = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma - z}\,\rho^{\rm eff}(\sigma)\,, \tag{13}$$

$$\bar{a}_s^{\rm eff}(s) = \frac{1}{\pi}\int_s^\infty \frac{d\sigma}{\sigma}\,\rho^{\rm eff}(\sigma)\,. \tag{14}$$

First, consider the contour representation (9) for the QCD contribution Δ_τ. By using Eq. (13) this formula can be rewritten as follows

$$\Delta_\tau \;=\; \frac{d_1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma}\,\rho^{\rm eff}(\sigma) \;-\; \frac{d_1}{\pi}\int_0^{M_\tau^2} \frac{d\sigma}{\sigma}\left(1-\frac{\sigma}{M_\tau^2}\right)^3\left(1+\frac{\sigma}{M_\tau^2}\right)\rho^{\rm eff}(\sigma)\,. \tag{15}$$

Of course, the same result (15) will be obtained if we substitute Eq. (14) into Eq. (8).

Due to the feature of universality, the first term in Eq. (15) can be expressed in terms of only the first β-function coefficient. To demonstrate this here let us use the following asymptotic representation for the effective coupling constant

$$a_{\rm PT}^{\rm eff}(q^2) \;=\; \frac{1}{\ell}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\alpha_{k,m}\frac{\ln^m\ell}{\ell^k}\,, \tag{16}$$

where $\ell \;=\; \ln(Q^2/\Lambda^2)$ and $\alpha_{0,0} \;=\; 1/\beta_0$. Calculating the effective spectral density as the discontinuity of the effective coupling (16), for the infrared limiting value of the analytic effective coupling constant we get, with $L = \ln\sigma/\Lambda^2$,

$$\begin{aligned}\bar{a}_{\rm APT}^{\rm eff}(0) \;&=\; \frac{1}{\pi}\int_{-\infty}^{\infty} dL\,\rho^{\rm eff}(L)\\ &=\; \frac{1}{\beta_0} + \sum_{k=1}^{\infty}\sum_{m=0}^{k}\alpha_{k,m}\,\Delta\,\bar{a}_{k,m}^{\rm eff}(0)\end{aligned} \tag{17}$$

with

$$\Delta\,\bar{a}_{k,m}^{\rm eff}(0) \;=\; \frac{1}{\pi}{\rm Im}\int_{-\infty}^{\infty} dL\,\frac{\ln^m(L-{\rm i}\pi)}{(L-{\rm i}\pi)^{k+1}}\,. \tag{18}$$

By repeatedly integrating by parts, it is now very easy to show that

$$\Delta\,\bar{a}_{k,m}^{\rm eff}(0) \;=\; \frac{m!}{k^m}\Delta\,\bar{a}_{k,0}^{\rm eff}(0) \;=\; 0. \tag{19}$$

Consider the 2-loop level, in which the perturbative effective running coupling constant is

$$a^{\rm eff}_{\rm PT}(q^2) \;=\; \bar{a}_{\rm PT}(q^2) \,+\, \frac{d_2}{d_1}\bar{a}^2_{\rm PT}(q^2)\,, \tag{20}$$

where for n_f active quarks $d_2 \,=\, 16\,(1.9857 - 0.1153 n_f)$ [11]. By using Eq. (20) we can obtain the corresponding effective spectral density. It is important to appreciate that in the framework of the APT approach there is very little sensitivity to the approximation used in solving the renormalization group equations. This fact is demonstrated in Fig. 1, where we plot three spectral densities. The solid line corresponds to the spectral density obtained from the exact quadrature of the 2-loop β function,

$$\beta_0 \ell \;=\; \int^{\bar{a}} \frac{dx}{\beta(x)} = \frac{1}{\bar{a}} - \beta_0 B_1 \ln\left(1+\frac{1}{\beta_0 B_1 \bar{a}}\right)\,, \tag{21}$$

where $B_1 = \beta_1/(\beta_0)^2$, and $\beta_1 = 102 - 38 n_f/3$ is the two-loop coefficient of the β-function.

The dashed line corresponds to the spectral density obtained from the asymptotic representation of the perturbative 2-loop running coupling,

$$\bar{a} \;=\; \frac{1}{\beta_0 \ell}\left(1 - B_1 \frac{\ln \ell}{\ell}\right)\,. \tag{22}$$

The dot-dashed line corresponds to the following perturbative running coupling

$$\bar{a}(Q^2) = \frac{1}{\beta_0 \ell + \beta_0 B_1 \ln(1+\ell/B_1)}\,, \tag{23}$$

This expression, which we will use for our numerical calculations, is the result of exact integration of the two-loop differential RG equation (21) solved by one iteration. For this case the spectral density which is associated with the running coupling (23) is

$$\varrho(\sigma) \;=\; \frac{1}{\beta_0}\,\frac{I(L)}{R^2(L) \,+\, I^2(L)}\,, \tag{24}$$

with

$$R(L) \;=\; L \,+\, B_1 \ln\sqrt{\left(1+\frac{L}{B_1}\right)^2 + \left(\frac{\pi}{B_1}\right)^2}\,, \tag{25}$$

$$I(L) \;=\; \pi \,+\, B_1 \arccos\frac{B_1 + L}{\sqrt{(B_1+L)^2 + \pi^2}}\,.$$

As a result, for the effective spectral density, we find

$$\rho^{\rm eff}(\sigma) \;=\; \varrho(\sigma) \,+\, \frac{1}{\beta_0^2}\frac{d_2}{d_1}\frac{2R(L)I(L)}{[R^2(L)+I^2(L)]^2}\,. \tag{26}$$

By substituting this formula into Eq. (15) and using the fact of universality for the first term we obtain the QCD correction for the inclusive decay rate of the τ lepton. For our

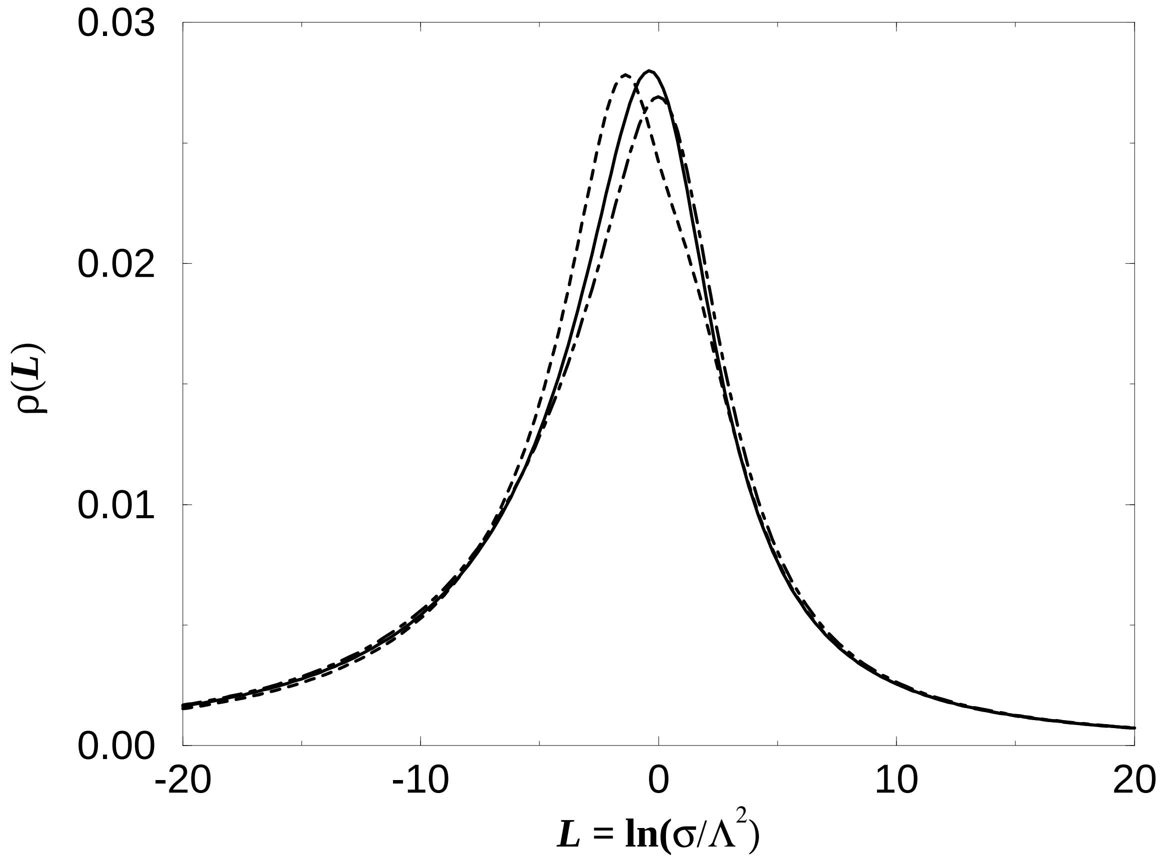

FIG. 1. *Plot of the spectral densities. The solid line refers to the exact quadrature of the RG equation, the dashed line refers to the asymptotic representation from the perturbative running coupling, and the dot-dashed line corresponds to solution obtained by one iteration.*

numerical estimations we will use the world average $R_\tau = 3.633 \pm 0.031$ [12], which leads to the following values of the APT running couplings: $\alpha_s(M_\tau^2) = 0.378 \pm 0.026$ in the timelike region and $\alpha(M_\tau^2) = 0.400 \pm 0.026$ in the spacelike region, respectively. The reader should carefully note that the values of the strong coupling constant in the timelike and the spacelike region do not agree, even though they are evaluated at the same scale. This difference could become significant experimentally as the precision of the measurements improves.

It should be stressed that these values of the coupling constants have been obtained with a value of the QCD scale parameter much larger[3] than that found in conventional perturbation theory: for three active flavors, $\Lambda_{(3)} = 935 \pm 159$ MeV. We have found that the value of Λ is very sensitive to the value of R_τ. For example, if we were to use a smaller value of $R_\tau = 3.559 \pm 0.035$ [14], we would obtain $\Lambda = 640 \pm 127$ MeV. Note here that,

[3]The same observation has been made in the one-loop level in [13].

as we have demonstrated, the conventional perturbative parameterization of the D-function is inconsistent with the required analyticity properties and it is not possible to write the contour integral (2) in order to extract $\Lambda_{\rm QCD}$. Nevertheless, in order to give the reader a feeling for the relation between the APT and PT values of Λ, we require that that the 2-loop D-functions be the same in both schemes at the τ scale, and obtain the relation between the two definitions of Λ shown in Fig. 2.

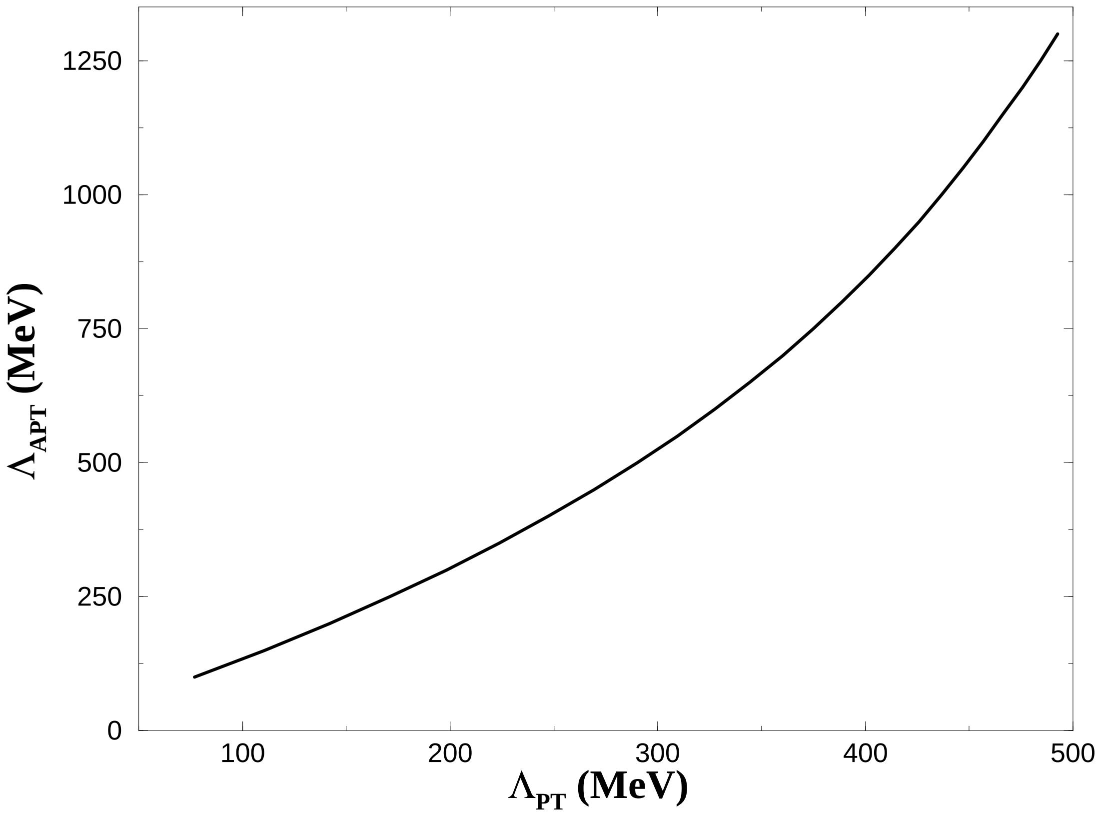

FIG. 2. $\Lambda_{\rm APT}$ *vs.* $\Lambda_{\rm PT}$ *at the* τ*-mass scale.*

It is possible to understand in a simple manner why in this approach the Λ parameter is so much larger than the PT value. To this end, we will use the approximate formula for the 2-loop APT coupling constant [4]:

$$\bar{a}(q^2) \simeq \bar{a}_{\rm PT}(q^2) + \frac{1}{2\beta_0}\frac{1}{1+q^2/\Lambda^2} + \frac{C_1}{\beta_0}\frac{\Lambda^2}{q^2}\,, \tag{27}$$

where, for three active quarks, $C_1 = 0.035$. The accuracy of this formula on the interval $2.5 < |q|/\Lambda < 3.5$ is about 0.5%, and Eq. (27) practically coincides with the exact formula for larger values of momentum. A key feature of this formula is that the perturbative and nonperturbative contributions are separated, as in the 1-loop case, which allows us to write the QCD correction to τ decay as the sum of perturbative and nonperturbative terms. The main part of the corresponding nonperturbative contribution is

$$\Delta_{\rm NP} = -\frac{d_1}{\beta_0}\left(1+2\frac{d_2}{d_1}\frac{1}{\beta_0}\right)\frac{\Lambda^2}{M_\tau^2}\,. \tag{28}$$

Due to the negative sign of this contribution, the perturbative term must be larger than in the case of PT (which implies a larger value of Λ), in order to obtain the same value of the QCD correction.

IV. CONCLUSION

We have considered the method of analytic perturbation theory and its application to the semileptonic decay of the τ lepton. It should be stressed that the principle of causality, the consequence of which are certain analytic properties of the running coupling constant, leads to the essential stability of the running coupling constant in the infrared region with respect to higher loop corrections. Here, the prime point is the universal value of the analytic coupling constant at $Q^2 = 0$, which does not depend on the experimental estimates of the QCD scale parameter, Λ, nor on the number of loops in which we approximate the running coupling constant. In other words, families of curves corresponding to different values of Λ and to different numbers of loops have the common point $\bar{\alpha}(Q^2 = 0) = 4\pi/\beta_0$, and are represented by a bundle of curves.

Moreover, this approach allows us to give a self-consistent definition of the running coupling constant in the timelike region. Thus, in this approach it is possible to give two equivalent expressions for the QCD correction to inclusive τ decay, either in terms of the timelike coupling, or in terms of the coupling defined for complex Euclidean momentum. This is not possible in conventional perturbation theory, due to the breaking of analyticity.

The fact that we are now able to define a consistent timelike coupling constant enables us to perform matching between regions with different numbers of active flavors in the physical region, where at least in the leading order the number of active quarks is an obvious fact associated with physical thresholds. As a result of this matching procedure the coupling constant in the Euclidean region will know about all physical thresholds through the dispersion relation.

The results of the analysis performed above demonstrate the importance of analyticity in the running coupling constant, not only from the fundamental point of view—a self-consistent theoretical description of τ decay—but also from the standpoint of giving a self-consistent description of the Q^2 evolution of the coupling constant and extracting the parameter $\Lambda_{\rm QCD}$ from the experimental data on τ decay.

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