Local and Global Casimir Energies in a Green's Function Approach

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The effects of quantum fluctuations in fields confined by background configurations may be simply and transparently computed using the Green's function approach pioneered by Schwinger. Not only can total energies and surface forces be computed in this way, but local energy densities, and in general, all components of the vacuum expectation value of the energy-momentum tensor may be calculated. For simple geometries this approach may be carried out exactly, which yields insight into what happens in less tractable situations. In this talk I will concentrate on the example of a scalar field in a circular cylindrical delta-function background. This situation is quite similar to that of a spherical delta-function background. The local energy density in these cases diverges as the surface of the background is approached, but these divergences are integrable. The total energy is finite in strong coupling, but in weak coupling a divergence occurs in third order. This universal feature is shown to reflect a divergence in the energy associated with the surface, the integrated local energy density within the shell itself, which surface energy should be removable by a process of renormalization.

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1. Casimir Energies for Spheres and Cylinders

The calculation of Casimir self-energies of material objects has become controversial,[1] although these concerns are nearly as old as the subject itself.[2–4] Although it appears possible to extract unique self-energies, they may be overwhelmed by terms which become divergent for ideal geometries.[5,6] Our attitude is that these terms may be uniquely removed by a process of renormalization, and that even the divergences revealed by heat-kernel methods[7,8] may be unambiguously isolated.

Table 1 summarizes the state of our knowledge concerning total Casimir self-energies for different simple configurations. The first row of the table refers to the Casimir energy of a perfectly conducting shell, either spherical or cylindrical, subject to electromagnetic fluctuations in the exterior and interior regions. The second row refers to the same results for a scalar field subject to Dirichlet boundary conditions on the surface. The remaining four rows describe small perturbations: Row 3 describes what happens for electromagnetic fluctuations when the interior of the sphere or cylinder is a dielectric having a permittivity ε differing slightly from the vacuum value of unity; Row 4 indicates the same when the speed of light is the same inside and outside the object, where $\xi=(\varepsilon'-\varepsilon)/(\varepsilon'+\varepsilon)$ in terms of the permittivity inside (ε) and outside (ε') the object; Row 5 shows the effect for a perfect conductor of a small ellipticity δe ($\pm$ refers to a prolate or oblate spheroid, respectively); and Row 6 refers to a δ-function potential (semitransparent shell) of strength λ, which will be described in this paper. In these four cases, what is shown in the table is the coefficient of the second-order term in the relevant small quantity. One of the ongoing challenges facing quantum field theorists attempting to understand the quantum vacuum is to understand the pattern of signs and zeroes manifested in the table.

Table 1. Casimir energy (E) for a sphere and Casimir energy per unit length ($\mathcal{E}$) for a cylinder, both of radius a. The signs indicate repulsion or attraction, respectively.

Type	$E_{\text{Sphere}}a$	$\mathcal{E}_{\text{Cylinder}}a^2$	References
EM	$+0.04618$	-0.01356	9, 10
D	$+0.002817$	$+0.0006148$	11, 12
$(\varepsilon-1)^2$	$+0.004767=\frac{23}{1536\pi}$	0	13, 14
ξ^2	$+0.04974=\frac{5}{32\pi}$	0	15, 16
δe^2	± 0.0009	0	17, 18
λ^2	$+0.009947=\frac{1}{32\pi}$	0	19, 20

In this talk, we will illustrate the ideas for the interesting case of a circular cylindrically symmetric annular potential. Most of the calculations will refer to a δ-function potential.

2. Green's Function

We consider a massless scalar field ϕ in a δ-cylinder background,

$$\mathcal{L}_{\text{int}}=-\frac{\lambda}{2a}\delta(r-a)\phi^2, \tag{1}$$

a being the radius of the "semitransparent" cylinder. We recall that the massive case was earlier considered by Scandurra.[21] Note that with this definition, λ is dimensionless. The time-Fourier transform of the Green's function,

$$G(x,x')=\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\mathcal{G}(\mathbf{r},\mathbf{r}'), \tag{2}$$

satisfies

$$\left[-\nabla^2-\omega^2+\frac{\lambda}{a}\delta(r-a)\right]\mathcal{G}(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}'). \tag{3}$$

Adopting cylindrical coordinates, we write

$$\mathcal{G}(\mathbf{r},\mathbf{r}')=\int\frac{dk}{2\pi}e^{ik(z-z')}\sum_{m=-\infty}^{\infty}\frac{1}{2\pi}e^{im(\varphi-\varphi')}g_m(r,r';k), \tag{4}$$

where the reduced Green's function satisfies

$$\left[-\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}+\kappa^2+\frac{m^2}{r^2}+\frac{\lambda}{a}\delta(r-a)\right]g_m(r,r';k)=\frac{1}{r}\delta(r-r'), \tag{5}$$

where $\kappa^2=k^2-\omega^2$. Let us immediately make a Euclidean rotation,

$$\omega\to i\zeta, \tag{6}$$

where ζ is real, so κ is always real and positive. Apart from the δ functions, Eq. (5) is the modified Bessel equation.

2.1. *Reduced Green's function*

Because of the Wronskian satisfied by the modified Bessel functions,

$$K_m(x)I_m'(x)-K_m'(x)I_m(x)=\frac{1}{x}, \tag{7}$$

we have the general solution to the Green's function equation (5) as long as $r\neq a$ to be

$$g_m(r,r';k)=I_m(\kappa r_<)K_m(\kappa r_>)+A(r')I_m(\kappa r)+B(r')K_m(\kappa r), \tag{8}$$

where A and B are arbitrary functions of r'. Now we incorporate the effect of the δ function at $r=a$ in the Green's function equation. It implies that g_m must be continuous at $r=a$, while it has a discontinuous derivative,

$$a\frac{d}{dr}g_m(r,r';k)\bigg|_{r=a-}^{r=a+}=\lambda g_m(a,r';k), \tag{9}$$

from which we rather immediately deduce the form of the Green's function inside and outside the cylinder:

$$r,r'<a:\quad g_m(r,r';k)=I_m(\kappa r_<)K_m(\kappa r_>)-\frac{\lambda K_m^2(\kappa a)}{1+\lambda I_m(\kappa a)K_m(\kappa a)}I_m(\kappa r)I_m(\kappa r'), \tag{10a}$$

$$r,r'>a:\quad g_m(r,r';k)=I_m(\kappa r_<)K_m(\kappa r_>)-\frac{\lambda I_m^2(\kappa a)}{1+\lambda I_m(\kappa a)K_m(\kappa a)}K_m(\kappa r)K_m(\kappa r'). \tag{10b}$$

Notice that in the limit $\lambda\to\infty$ we recover the Dirichlet cylinder result, that is, that g_m vanishes at $r=a$.

3. Pressure and Energy

The easiest way to calculate the total energy is to compute the pressure on the cylindrical walls due to the quantum fluctuations in the field. This may be computed, at the one-loop level, from the vacuum expectation value of the stress tensor,

$$\langle T^{\mu\nu}\rangle=\left(\partial^\mu\partial'^\nu-\frac{1}{2}g^{\mu\nu}\partial^\lambda\partial'_\lambda\right)\frac{1}{i}G(x,x')\bigg|_{x=x'}-\xi(\partial^\mu\partial^\nu-g^{\mu\nu}\partial^2)\frac{1}{i}G(x,x). \tag{11}$$

Here we have included the conformal parameter ξ, which is equal to 1/6 for the conformal stress tensor. The conformal term does not contribute to the radial-radial component of the stress tensor, however, because then only transverse and time derivatives act on $G(x,x)$, which depends only on r. The discontinuity of the expectation value of the radial-radial component of the stress tensor is the pressure on the cylindrical wall:

$$\begin{aligned}P &= \langle T_{rr}\rangle_{\rm in} - \langle T_{rr}\rangle_{\rm out}\\ &= -\frac{1}{16\pi^3}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}d\zeta\frac{\lambda\kappa^2}{1+\lambda I_m(\kappa a)K_m(\kappa a)}\\ &\qquad\times\left[K_m^2(\kappa a)I_m'^2(\kappa a)-I_m^2(\kappa a)K_m'^2(\kappa a)\right]\\ &= -\frac{1}{16\pi^3}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}d\zeta\frac{\kappa}{a}\frac{d}{d\kappa a}\ln\left[1+\lambda I_m(\kappa a)K_m(\kappa a)\right],\end{aligned} \tag{12}$$

where we've again used the Wronskian (7). Regarding ka and ζa as the two Cartesian components of a two-dimensional vector, with magnitude $x\equiv\kappa a=\sqrt{k^2a^2+\zeta^2a^2}$, we get the stress on the cylinder per unit length to be

$$\mathcal{S} = 2\pi a P = -\frac{1}{4\pi a^3}\int_0^\infty dx\,x^2\sum_{m=-\infty}^{\infty}\frac{d}{dx}\ln\left[1+\lambda I_m(x)K_m(x)\right], \tag{13}$$

implying the Dirichlet limit as $\lambda\to\infty$. By integrating $\mathcal{S}=-\frac{\partial}{\partial a}\mathcal{E}$, we obtain the energy per unit length

$$\mathcal{E} = -\frac{1}{8\pi a^2}\int_0^\infty dx\,x^2\sum_{m=-\infty}^{\infty}\frac{d}{dx}\ln\left[1+\lambda I_m(x)K_m(x)\right]. \tag{14}$$

This formal expression will be regulated, and evaluated in weak and strong coupling, in the following.

3.1. *Energy*

Alternatively, we may compute the energy directly from the general formula[22]

$$E = \frac{1}{2i}\int(d\mathbf{r})\int\frac{d\omega}{2\pi}2\omega^2\mathcal{G}(\mathbf{r},\mathbf{r}). \tag{15}$$

To evaluate the energy in this case, we need the indefinite integrals

$$\int_0^x dy\,y\,I_m^2(y) = \frac{1}{2}\left[(x^2+m^2)I_m^2(x)-x^2I_m'^2\right], \tag{16a}$$

$$\int_x^\infty dy\,y\,K_m^2(y) = -\frac{1}{2}\left[(x^2+m^2)K_m^2(x)-x^2K_m'^2\right]. \tag{16b}$$

When we insert the above construction (10) of the Green's function, and perform the integrals as indicated over the regions interior and exterior to the cylinder, we obtain

$$\mathcal{E} = -\frac{a^2}{8\pi^2}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}d\zeta\int_{-\infty}^{\infty}dk\,\zeta^2\frac{1}{x}\frac{d}{dx}\ln\left[1+\lambda I_m(x)K_m(x)\right]. \tag{17}$$

Again we regard the two integrals as over Cartesian coordinates, and replace the integral measure by

$$\int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} dk\, \zeta^2 = \pi \int_0^{\infty} d\kappa\, \kappa^3. \tag{18}$$

The result for the energy (14) immediately follows.

4. Weak-coupling Evaluation

Suppose we regard λ as a small parameter, so let us expand the energy (14) in powers of λ. The first term is

$$\mathcal{E}^{(1)} = -\frac{\lambda}{8\pi a^2} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dx\, x^2 \frac{d}{dx} K_m(x) I_m(x). \tag{19}$$

The addition theorem for the modified Bessel functions is

$$K_0(kP) = \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} K_m(k\rho) I_m(k\rho'), \quad \rho > \rho', \tag{20}$$

where $P = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')}$. If this is extrapolated to the limit $\rho' = \rho$ we conclude that the sum of the Bessel functions appearing in $\mathcal{E}^{(1)}$ is $K_0(0)$, that is, a constant, so there is no first-order contribution to the energy, $\mathcal{E}^{(1)} = 0$.

4.1. *Regulated numerical evaluation of* $\mathcal{E}^{(1)}$

Given that the above argument evidently formally omits divergent terms, it may be more satisfactory to offer a regulated numerical evaluation of $\mathcal{E}^{(1)}$. We can very efficiently do so using the uniform asymptotic expansions ($m \to \infty$):

$$I_m(x) \sim \sqrt{\frac{t}{2\pi m}} e^{m\eta} \left(1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{m^k}\right), \tag{21a}$$

$$K_m(x) \sim \sqrt{\frac{\pi t}{2m}} e^{-m\eta} \left(1 + \sum_{k=1}^{\infty} (-1)^k \frac{u_k(t)}{m^k}\right), \tag{21b}$$

where $x = mz$, $t = 1/\sqrt{1+z^2}$, and $\frac{d\eta}{dz} = \frac{1}{zt}$. The polynomials in t appearing here are generated by

$$u_0(t) = 1, \quad u_k(t) = \frac{1}{2} t^2 (1 - t^2) u'_{k-1}(t) + \int_0^t ds \frac{1 - 5s^2}{8} u_{k-1}(s). \tag{22}$$

Thus the asymptotic behavior of the products of Bessel functions appearing in Eq. (19) is obtained from

$$I_m^2(x) K_m^2(x) \sim \frac{t^2}{4m^2} \left(1 + \sum_{k=1}^{\infty} \frac{r_k(t)}{m^{2k}}\right). \tag{23}$$

The first three polynomials occurring here are

$$r_1(t) = \frac{t^2}{4}(1 - 6t^2 + 5t^4), \tag{24a}$$

$$r_2(t) = \frac{t^4}{16}(7 - 148t^2 + 554t^4 - 708t^6 + 295t^8), \tag{24b}$$

$$r_3(t) = \frac{t^6}{16}(36 - 1666t^2 + 13775t^4 - 44272t^6 + 67162t^8 - 48510t^{10} + 13475t^{12}). \tag{24c}$$

We regulate the sum and integral by inserting an exponential cutoff, $\delta \to 0+$:

$$\mathcal{E}^{(1)} = -\frac{\lambda}{4\pi a^2} {\sum_{m=0}^{\infty}}' \int_0^\infty dx\, x^2 \frac{d}{dx} I_m(x) K_m(x) e^{-x\delta}, \tag{25}$$

where the prime on the summation sign means that the $m = 0$ term is counted with one-half weight. We break up this expression into five parts,

$$\mathcal{E}^{(1)} = -\frac{\lambda}{8\pi a^2}(\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}). \tag{26}$$

The first term is the $m = 0$ contribution, suitably subtracted to make it convergent (so the convergence factor may be omitted),

$$\mathrm{I} = \int_0^\infty dx\, x^2 \frac{d}{dx}\left[I_0(x)K_0(x) - \frac{1}{2\sqrt{1+x^2}}\right] = -1. \tag{27}$$

The second term is the above subtraction,

$$\mathrm{II} = \frac{1}{2}\int_0^\infty dx\, x^2 \left(\frac{d}{dx}\frac{1}{\sqrt{1+x^2}}\right) e^{-x\delta} \sim -\frac{1}{2\delta} + 1, \tag{28}$$

as may be verified by breaking the integral in two parts at Λ, $1 \ll \Lambda \ll 1/\delta$. The third term is the sum over the mth Bessel function with the two leading asymptotic approximants in Eq. (23) subtracted:

$$\mathrm{III} = 2\sum_{m=1}^{\infty}\int_0^\infty dx\, x^2 \frac{d}{dx}\left[I_m(x)K_m(x) - \frac{t}{2m}\left(1 + \frac{t^2}{8m^2}(1 - 6t^2 + 5t^4)\right)\right] = 0. \tag{29}$$

Numerically, each term in the sum seems to be zero to machine accuracy. This is verified by computing the higher-order terms in that expansion, in terms of the polynomials in Eq. (24):

$$I_m(x)K_m(x) - \frac{t}{2m}\left(1 + \frac{t^2}{8m^2}(1 - 6t^2 + 5t^4)\right) \sim \frac{t}{4m^5}\left[r_2(t) - \frac{1}{4}r_1^2(t)\right] + \frac{t}{4m^7}\left[r_3(t) - \frac{1}{2}r_1(t)r_2(t) + \frac{1}{8}r_1^3(t)\right] + \dots, \tag{30}$$

which terms are easily seen to integrate to zero. The fourth term is the leading subtraction which appeared in the third term:

$$\mathrm{IV} = \sum_{m=1}^{\infty} m \int_0^\infty dz\, z^2 \left(\frac{d}{dz}t\right) e^{-mz\delta}. \tag{31}$$

If we first carry out the sum on m we obtain

$$\mathrm{IV} = -\frac{1}{4}\int_0^\infty dz\, z^3 \frac{1}{(1+z^2)^{3/2}}\frac{1}{\sinh^2 z\delta/2} \sim -\frac{1}{\delta^2}+\frac{1}{2\delta}-\frac{1}{6}, \tag{32}$$

as verified by breaking up the integral. The final term, due to the subleading subtraction, if unregulated, is the form of infinity times zero:

$$\mathrm{V} = \frac{1}{8}\sum_{m=1}^\infty \frac{1}{m}\int_0^\infty dz\, z^2 \frac{d}{dz}(t^3 - 6t^5 + 5t^7)e^{-mz\delta}. \tag{33}$$

Here the sum on m gives

$$\sum_{m=1}^\infty \frac{1}{m} e^{-mz\delta} = -\ln\left(1 - e^{-z\delta}\right), \tag{34}$$

and so we can write

$$\mathrm{V} = \frac{1}{16}\frac{d}{d\alpha}\int_0^1 du\,(1-u)^\alpha u^{-2-\alpha}(u^{3/2} - 6u^{5/2} + 5u^{7/2})\bigg|_{\alpha=0} = \frac{1}{6}. \tag{35}$$

Adding together these five terms we obtain

$$\mathcal{E}^{(1)} = \frac{\lambda}{8\pi a^2\delta^2} + 0, \tag{36}$$

that is, the $1/\delta$ and constant terms cancel. The remaining divergence may be interpreted as an irrelevant constant, since $\delta = \tau/a$, τ being regarded as a point-splitting parameter. This thus agrees with the result stated at the beginning of this section.

4.2. λ^2 *term*

We can proceed the same way to evaluate the second-order contribution to Eq. (14),

$$\mathcal{E}^{(2)} = \frac{\lambda^2}{16\pi a^2}\int_0^\infty dx\, x^2\,\frac{d}{dx}\sum_{m=-\infty}^\infty I_m^2(x)K_m^2(x). \tag{37}$$

By squaring the sum rule (20), and again taking the formal singular limit $\rho' \to \rho$, we evaluate the sum over Bessel functions appearing here as

$$\sum_{m=-\infty}^\infty I_m^2(x)K_m^2(x) = \int_0^{2\pi}\frac{d\varphi}{2\pi}K_0^2(2x\sin\varphi/2). \tag{38}$$

Then changing the order of integration, we can write the second-order energy as

$$\mathcal{E}^{(2)} = -\frac{\lambda^2}{64\pi^2 a^2}\int_0^{2\pi}\frac{d\varphi}{\sin^2\varphi/2}\int_0^\infty dz\, z\, K_0^2(z), \tag{39}$$

where the Bessel-function integral has the value 1/2. However, the integral over φ is divergent. We interpret this integral by adopting an analytic regularization based on the integral ($\mathrm{Re}\, s > -1$)

$$\int_0^{2\pi} d\varphi \left(\sin\frac{\varphi}{2}\right)^s = \frac{2\sqrt{\pi}\,\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(1+\frac{s}{2}\right)}. \tag{40}$$

Taking the right-side of this equation to define the φ integral for all s, we conclude that the φ integral, and hence the second-order energy $\mathcal{E}^{(2)}$, is zero.

The vanishing of the energy in order λ and λ^2 may be given a quite rigorous derivation in the zeta-function approach to Casimir energies—See Ref. 20.

4.2.1. *Alternative numerical evaluation*

Again we provide a numerical approach to bolster our argument. Subtracting and adding the leading asymptotic behaviors, we now write the second-order energy as $(z = x/m)$

$$\begin{aligned}\mathcal{E}^{(2)} = -\frac{\lambda^2}{8\pi a^2}\Biggl\{ \int_0^\infty dx\, x\left[I_0^2(x)K_0^2(x) - \frac{1}{4(1+x^2)}\right] \\ + \frac{1}{2}\lim_{s\to 0}\sum_{m=0}^{\infty}{}' m^{-s}\int_0^\infty dz \frac{z^{1-s}}{1+z^2} + 2\int_0^2 dz\, z\frac{t^2}{4}\sum_{m=1}^{\infty}\sum_{k=1}^{3}\frac{r_k(t)}{m^{2k}} \\ + 2\sum_{m=1}^{\infty}\int_0^\infty dx\, x\left[I_m^2(x)K_m^2(x) - \frac{t^2}{4m^2}\left(1+\sum_{k=1}^{3}\frac{r_l(t)}{m^{2k}}\right)\right]\Biggr\}. \end{aligned} \tag{41}$$

The successive terms are evaluated as

$$\begin{aligned}\mathcal{E}^{(2)} \approx -\frac{\lambda^2}{8\pi a^2}\Biggl[\frac{1}{4}(\gamma+\ln 4) - \frac{1}{4}\ln 2\pi - \frac{\zeta(2)}{48} + \frac{7\zeta(4)}{1920} - \frac{31\zeta(6)}{16128} \\ +0.000864 + 0.000006\Biggr] = -\frac{\lambda^2}{8\pi a^2}(0.000000),\end{aligned} \tag{42}$$

where in the last term in the energy (41) only the $m = 1$ and 2 terms are significant. Therefore, we have demonstrated numerically that the energy in order λ^2 is zero to an accuracy of better than 10^{-6}.

4.2.2. *Exponential regulator*

The astute listener will note that we used a standard, but possibly questionable, analytic regularization in defining the second term in energy above. Alternatively, as in Sec. 4.1 we could insert there an exponential regulator in each integral of $e^{-x\delta}$, with δ to be taken to zero at the end of the calculation. For $m \neq 0$ x becomes mz, and then the sum on m becomes

$$\sum_{m=1}^{\infty} e^{-mz\delta} = \frac{1}{e^{z\delta}-1}. \tag{43}$$

Then when we carry out the integral over z we obtain for that term

$$\frac{\pi}{8\delta} - \frac{1}{4}\ln 2\pi. \tag{44}$$

Thus we obtain the same finite part as above, but in addition an explicitly divergent term

$$\mathcal{E}^{(2)}_{\text{div}} = -\frac{\lambda^2}{64a^2\delta}. \tag{45}$$

Again, if we think of the cutoff in terms of a vanishing proper time τ, $\delta = \tau/a$, this divergent term is proportional to $1/a$, so the divergence in the energy goes like L/a, if L is the (very large) length of the cylinder. This is of the form of the *shape divergence* encountered in Ref. 14.

4.3. *Divergence in $O(\lambda^3)$*

Although the first two orders in λ identically vanish, a divergence in the energy (14) does occur in $O(\lambda^3)$.

$$\begin{aligned}\mathcal{E}^{(3)} &= -\frac{1}{8\pi a^2}\sum_{m=-\infty}^{\infty}\int_0^\infty dx\, x^{2-s}\frac{d}{dx}\frac{1}{3}\lambda^3 K_m^3(x) I_m^3(x)\\ &\sim \frac{\lambda^3}{96\pi a^2 s}, \quad s \to 0. \end{aligned}\tag{46}$$

That such a divergence does occur generically in third order was proved in Ref. 20, using heat-kernel techniques. As we shall see, this divergence entirely arises from the *surface energy.*

5. Strong Coupling

The strong-coupling limit of the energy (14), that is, the Casimir energy of a Dirichlet cylinder,

$$\mathcal{E}^D = -\frac{1}{8\pi a^2}\sum_{m=-\infty}^{\infty}\int_0^\infty dx\, x^2\frac{d}{dx}\ln I_m(x)K_m(x), \tag{47}$$

was worked out to high accuracy by Gosdzinski and Romeo,[12]

$$\mathcal{E}^D = \frac{0.000614794033}{a^2}. \tag{48}$$

It was later redone with less accuracy by Nesterenko and Pirozhenko.[23] For completeness, let us sketch the evaluation here. Again subtracting and adding the leading asymptotics, we find for the energy per unit length

$$\begin{aligned}\mathcal{E}^D = -\frac{1}{8\pi a^2}\bigg\{ &-2\int_0^\infty dx\,x\left[\ln\left(2xI_0(x)K_0(x)\right)-\frac{1}{8}\frac{1}{1+x^2}\right]\\ &+2\sum_{m=1}^\infty\int_0^\infty dx\,x^2\frac{d}{dx}\left[\ln\left(2xI_m(x)K_m(x)\right)-\ln\left(\frac{xt}{m}\right)-\frac{1}{2}\frac{r_1(t)}{m^2}\right]\\ &-2{\sum_{m=0}^\infty}'\int_0^\infty dx\,x^2\frac{d}{dx}\ln 2x+2\sum_{m=1}^\infty\int_0^\infty dx\,x^2\frac{d}{dx}\ln xt\\ &+\sum_{m=1}^\infty\int_0^\infty dx\,x^2\frac{d}{dx}\left[\frac{r_1(t)}{m^2}-\frac{1}{4}\frac{1}{1+x^2}\right]-\frac{1}{2}{\sum_{m=0}^\infty}'\int_0^\infty dx\frac{x}{1+x^2}\bigg\}. \qquad (49)\end{aligned}$$

In the first two terms we have subtracted the leading asymptotic behavior so the resulting integrals are convergent. Those terms are restored in the fourth, fifth, and sixth terms. The most divergent part of the Bessel functions are removed by the insertion of $2x$ in the corresponding integral, and its removal in the third term. (Elsewhere, such terms have been referred to as “contact terms.”) The terms involving Bessel functions are evaluated numerically, where it is observed that the asymptotic value of the summand (for large m) in the second term is $1/32m^2$. The fourth term is evaluated by writing it as

$$2\lim_{s\to 0}\sum_{m=1}^\infty m^{2-s}\int_0^\infty dz\frac{z^{1-s}}{1+z^2}=2\zeta'(-2)=-\frac{\zeta(3)}{2\pi^2}, \qquad (50)$$

while the same argument, as anticipated, shows that the third “contact” term is zero.[a] The sixth term is

$$-\frac{1}{2}\lim_{s\to 0}\left[\zeta(s)+\frac{1}{2}\right]\frac{1}{s}=\frac{1}{4}\ln 2\pi. \qquad (51)$$

The fifth term is elementary. The result then is

$$\begin{aligned}\mathcal{E}^D &= \frac{1}{4\pi a^2}(0.010963-0.0227032+0+0.0304485+0.21875-0.229735)\\ &=\frac{0.0006146}{a^2}, \qquad (52)\end{aligned}$$

which agrees with Eq. (48) to the fourth significant figure.

5.1. *Exponential regulator*

As in the weak-coupling calculation, it may seem more satisfactory to insert an exponential regulator rather than use analytic regularization. Now it is the third, fourth, and sixth terms in the above expression that must be treated. The latter is

[a]This argument is a bit suspect, since the analytic continuation that defines the integrals has no common region of existence. Thus the argument in the following subsection may be preferable.

just the negative of the term (44) encountered in weak coupling. We can combine the third and fourth terms to give

$$-\frac{1}{\delta^2}+\frac{2}{\delta^2}\int_0^\infty \frac{dz\,z^3}{z^2+\delta^2}\frac{d^2}{dz^2}\frac{1}{e^z-1}. \tag{53}$$

The latter integral may be evaluated by writing it as an integral along the entire z axis, and closing the contour in the upper half plane, thereby encircling the poles at $i\delta$ and at $2in\pi$, where n is a positive integer. The residue theorem then gives for that integral

$$-\frac{2\pi}{\delta^3}-\frac{\zeta(3)}{2\pi^2}, \tag{54}$$

so once again, comparing with Eq. (50), we obtain the same finite part as in Eq. (52).

In this way of proceeding, then, in addition to the finite part found before in Eq. (52), we obtain divergent terms

$$\mathcal{E}^D_{\rm div}=\frac{1}{64a^2\delta}+\frac{1}{8\pi a^2\delta^2}+\frac{1}{4a^2\delta^3}, \tag{55}$$

which, with the previous interpretation for δ, implies terms in the energy proportional to L/a (shape), L (length), and aL (area), respectively, and are therefore *renormalizable*. Had a logarithmic divergence occurred (as does occur in weak coupling in $O(\lambda^3)$) such a renormalization would be impossible. However, see below!

6. Local Energy Density

We compute the energy density from the stress tensor (11), or

$$\begin{aligned}\langle T^{00}\rangle &= \frac{1}{2i}\left(\partial^0\partial^{0\prime}+\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}'\right)G(x,x')\bigg|_{x'=x}-\frac{\xi}{i}\nabla^2 G(x,x)\\ &=\frac{1}{16\pi^3 i}\int_{-\infty}^\infty dk\int_{-\infty}^\infty d\omega\sum_{m=-\infty}^\infty\left[\left(\omega^2+k^2+\frac{m^2}{r^2}+\partial_r\partial_{r'}\right)g(r,r')\bigg|_{r'=r}\right.\\ &\qquad\left.-2\xi\frac{1}{r}\partial_r r\partial_r g(r,r)\right].\end{aligned} \tag{56}$$

We omit the free part of the Green's function (10), since that corresponds to the vacuum energy in the absence of the cylinder. When we insert the remainder of the Green's function, we obtain the following expression for the energy density outside the cylindrical shell:

$$\begin{aligned}u(r)=&-\frac{\lambda}{16\pi^3}\int_{-\infty}^\infty d\zeta\int_{-\infty}^\infty dk\sum_{m=-\infty}^\infty\frac{I_m^2(\kappa a)}{1+\lambda I_m(\kappa a)K_m(\kappa a)}\\ &\times\left[\left(2\omega^2+\kappa^2+\frac{m^2}{r^2}\right)K_m^2(\kappa r)+\kappa^2 K_m'^2(\kappa r)\right.\\ &\left.\quad-2\xi\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}K_m^2(\kappa r)\right],\qquad r>a.\end{aligned} \tag{57}$$

The factor in square brackets can be easily seen to be, from the modified Bessel equation,

$$2\omega^2 K_m^2(\kappa r) + \frac{1-4\xi}{2}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}K_m^2(\kappa r). \tag{58}$$

For the interior region, $r < a$, we have the corresponding expression for the energy density with $I_m \leftrightarrow K_m$.

6.1. *Total and surface energy*

We first need to verify that we recover the expression for the energy found before. So let us integrate the above expression over the region exterior of the cylinder, and the corresponding interior expression over the inside region. The second term in Eq. (58) is a total derivative, while the first may be integrated according to the integrals given in Eq. (16). In fact that term is exactly that evaluated above. The result is

$$\begin{aligned}\int(d\mathbf{r})\,u(r) &= -\frac{1}{8\pi a^2}\sum_{m=-\infty}^{\infty}\int_0^\infty dx\,x^2\frac{d}{dx}\ln\left[1+\lambda I_m(x)K_m(x)\right]\\ &\quad-(1-4\xi)\frac{\lambda}{4\pi a^2}\int_0^\infty dx\,x\sum_{m=-\infty}^{\infty}\frac{I_m(x)K_m(x)}{1+\lambda I_m(x)K_m(x)}.\end{aligned} \tag{59}$$

The first term is the total energy (14), but what do we make of the second term? In strong coupling, it would represent a constant that should have no physical significance (a contact term—it is independent of a if we revert to the physical variable κ as the integration variable).

In general, however, there is another contribution to the total energy, residing precisely on the singular surface. This surface energy is given in general by[22,24–28]

$$\mathfrak{E} = -\frac{1-4\xi}{2i}\oint_S d\mathbf{S}\cdot\boldsymbol{\nabla}G(x,x')\bigg|_{x'=x}, \tag{60}$$

which turns out to be the negative of the second term in $\int(d\mathbf{r})\,u(r)$ given in Eq. (59). This is an example of the general theorem

$$\int(d\mathbf{r})\,u(\mathbf{r}) + \mathfrak{E} = E, \tag{61}$$

that is, the total energy E is the sum of the integrated local energy density and the surface energy. A consequence of this theorem is that the total energy, unlike the local energy density, is independent of the conformal parameter ξ.

6.2. *Surface divergences*

We now turn to an examination of the behavior of the local energy density as r approaches a from outside the cylinder. To do this we use the uniform asymptotic expansion (21). Let us begin by considering the strong-coupling limit, a Dirichlet cylinder. If we stop with only the leading asymptotic behavior, we obtain the expression ($z = \kappa r/m$)

$$u(r) \sim -\frac{1}{8\pi^3}\int_0^\infty d\kappa\,\kappa \sum_{m=-\infty}^{\infty} e^{-m\chi}\left[-\kappa^2\frac{\pi t}{2m} + 2(1-4\xi)\kappa^2\frac{\pi}{2mt}\frac{1}{z^2}\right], \qquad (\lambda\to\infty) \tag{62}$$

where

$$\chi = -2\left[\eta(z) - \eta\left(z\frac{a}{r}\right)\right], \tag{63}$$

and we have carried out the “angular” integral as in Eq. (18). Here we ignore the difference between r and a except in the exponent, and we now replace κ by mz/a. Close to the surface,

$$\chi \sim \frac{2}{t}\frac{r-a}{r}, \tag{64}$$

and we carry out the sum over m according to

$$2\sum_{m=1}^{\infty} m^3 e^{-m\chi} \sim -2\frac{d^3}{d\chi^3}\frac{1}{\chi} = \frac{12}{\chi^4} \sim \frac{3}{4}\frac{t^4 r^4}{(r-a)^4}. \tag{65}$$

Then the energy density behaves, as $r \to a+$,

$$u(r) \sim -\frac{1}{16\pi^2}\frac{1}{(r-a)^4}(1-6\xi). \tag{66}$$

This is the universal surface divergence first discovered by Deutsch and Candelas.[2] It therefore occurs, with precisely the same numerical coefficient, near a Dirichlet plate[19] or a Dirichlet sphere.[29] It is utterly without physical significance (in the absence of gravity), and may be eliminated with the conformal choice for the parameter ξ, $\xi = 1/6$.

6.3. *Conformal surface divergence*

We will henceforth make this conformal choice. Then the leading divergence depends upon the curvature. This was also worked out by Deutsch and Candelas;[2] for the case of a cylinder, that result is

$$u(r) \sim \frac{1}{720\pi^2}\frac{1}{r(r-a)^3}, \quad r \to a+, \tag{67}$$

exactly 1/2 that for a Dirichlet sphere of radius a. To get this result, we keep the $1/m$ corrections in the uniform asymptotic expansion, and the next term in χ:

$$\chi \sim \frac{2}{t}\frac{r-a}{r} + t\left(\frac{r-a}{r}\right)^2. \tag{68}$$

6.4. *Weak coupling*

Let us now expand the energy density (57) for small coupling,

$$u(r) = -\frac{\lambda}{16\pi^3}\int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} dk \sum_{m=-\infty}^{\infty} I_m^2(\kappa a) \sum_{n=0}^{\infty} (-\lambda)^n I_m^n(\kappa a) K_m^n(\kappa a)$$
$$\times\left\{ \left[-\kappa^2 + (1-4\xi)\left(\kappa^2 + \frac{m^2}{r}\right)\right] K_m^2(\kappa r) + (1-4\xi)\kappa^2 K_m'^2(\kappa r)\right\}. \quad (69)$$

If we again use the leading uniform asymptotic expansions for the Bessel functions we obtain the expression for the leading behavior of the term of order λ^n,

$$u^{(n)}(r) \sim \frac{1}{8\pi^2 r^4}\left(-\frac{\lambda}{2}\right)^n \int_0^\infty dz\, z \sum_{m=1}^{\infty} m^{3-n} e^{-m\chi} t^{n-1}(t^2 + 1 - 8\xi). \quad (70)$$

The sum on m is asymptotic to

$$\sum_{m=1}^{\infty} m^{3-n} e^{-m\chi} \sim (3-n)!\left(\frac{tr}{2(r-a)}\right)^{4-n}, \quad r \to a+, \quad (71)$$

so the most singular behavior of the order λ^n term is, as $r \to a+$,

$$u^{(n)}(r) \sim (-\lambda)^n \frac{(3-n)!\,(1-6\xi)}{96\pi^2 r^n (r-a)^{4-n}}. \quad (72)$$

This is exactly the result found for the weak-coupling limit for a δ-sphere[29] and for a δ-plane,[22] so this is a universal result, without physical significance. It may be made to vanish by choosing the conformal value $\xi = 1/6$.

6.5. *Conformal weak coupling*

With this conformal choice, once again we must expand to higher order. Besides the corrections noted in Sec. 6.3, we also need

$$\tilde{t} \equiv t(za/r) \sim t + (t - t^3)\frac{r-a}{r}, \qquad r \to a, \quad (73)$$

Then a quite simple calculation gives

$$u^{(n)} \sim (-\lambda)^n \frac{(n-1)(n+2)\Gamma(3-n)}{2880\pi^2 r^{n+1}(r-a)^{3-n}}, \quad r \to a+, \quad (74)$$

which is analytically continued from the region $1 \le \mathrm{Re}\, n < 3$. Remarkably, this is exactly one-half the result found in the same weak-coupling expansion for the leading conformal divergence outside a sphere.[29] Therefore, like the strong-coupling result, this limit is universal, depending on the sum of the principal curvatures of the interface. Note this vanishes for $n = 1$, so in every case this divergence is integrable.

7. Cylindrical Shell of Finite Thickness

We now regard the shell (annulus) to have a finite thickness δ. We consider the potential

$$\mathcal{L}_{\rm int} = -\frac{\lambda}{2a}\phi^2\sigma(r), \tag{75}$$

where

$$\sigma(r) = \begin{cases} 0, & r < a_-, \\ h, & a_- < r < a_+, \\ 0, & a_+ < r. \end{cases} \tag{76}$$

Here $a_\pm = a \pm \delta/2$, and we set $h\delta = 1$. In the limit as $\delta \to 0$ we recover the δ-function potential. As for the sphere[29] it is straightforward to find the Green's function for this potential. In fact, the result may be obtained from the reduced Green's function given in Ref. 29 by an evident substitution. Here, we content ourselves by stating the result for the Green's function in the region of the annulus, $a_- < r, r' < a_+$:

$$\begin{aligned} g_m(r,r') &= I_m(\kappa' r_<)K_m(\kappa' r_>) + A I_m(\kappa' r) I_m(\kappa' r') \\ &\quad + B[I_m(\kappa' r)K_m(\kappa' r') + K_m(\kappa' r)I_m(\kappa' r')] + C K_m(\kappa' r)K_m(\kappa' r'), \end{aligned} \tag{77}$$

where $\kappa' = \sqrt{\kappa^2 + \lambda h/a}$. The coefficients appearing here are

$$\begin{aligned} A = -\frac{1}{\Xi}&[\kappa I_m'(\kappa a_-)K_m(\kappa' a_-) - \kappa' I_m(\kappa a_-)K_m'(\kappa' a_-)] \\ &\times[\kappa K_m'(\kappa a_+)K_m(\kappa' a_+) - \kappa' K_m(\kappa a_+)K_m'(\kappa' a_+)], \end{aligned} \tag{78a}$$

$$\begin{aligned} B = \frac{1}{\Xi}&[\kappa I_m'(\kappa a_-)I_m(\kappa' a_-) - \kappa' I_m(\kappa a_-)I_m'(\kappa' a_-)] \\ &\times[\kappa K_m'(\kappa a_+)K_m(\kappa' a_+) - \kappa' K_m(\kappa a_+)K_m'(\kappa' a_+)], \end{aligned} \tag{78b}$$

$$\begin{aligned} C = -\frac{1}{\Xi}&[\kappa I_m'(\kappa a_-)I_m(\kappa' a_-) - \kappa' I_m(\kappa a_-)I_m'(\kappa' a_-)] \\ &\times[\kappa K_m'(\kappa a_+)I_m(\kappa' a_+) - \kappa' K_m(\kappa a_+)I_m'(\kappa' a_+)], \end{aligned} \tag{78c}$$

where the denominator is

$$\begin{aligned} \Xi = {}&[\kappa I_m'(\kappa a_-)K_m(\kappa' a_-) - \kappa' I_m(\kappa a_-)K_m'(\kappa' a_-)] \\ &\times[\kappa K_m'(\kappa a_+)I_m(\kappa' a_+) - \kappa' K_m(\kappa a_+)I_m'(\kappa' a_+)] \\ &-[\kappa I_m'(\kappa a_-)I_m(\kappa' a_-) - \kappa' I_m(\kappa a_-)I_m'(\kappa' a_-)] \\ &\times[\kappa K_m'(\kappa a_+)K_m(\kappa' a_+) - \kappa' K_m(\kappa a_+)K_m'(\kappa' a_+)]. \end{aligned} \tag{79}$$

7.1. *Energy within the shell*

The general expression for the energy density within the shell is given in terms of these coefficients by

$$u(r) = \frac{1}{8\pi^2}\int_0^\infty d\kappa\,\kappa\left[-\kappa^2 + (1-4\xi)\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\right] \times \sum_{m=-\infty}^{\infty}[AI_m^2(\kappa' r) + CK_m^2(\kappa' r) + 2BK_m(\kappa' r)I_m(\kappa' r)]. \tag{80}$$

7.2. *Leading surface divergence*

The above expressions are somewhat formidable. Therefore, to isolate the most divergent structure, we replace the Bessel functions by the leading uniform asymptotic behavior (21). A simple calculation implies

$$A \sim \frac{t_+ - t'_+}{t_+ + t'_+}e^{-2m\eta'_+}, \tag{81a}$$

$$B \sim \frac{t_+ - t'_+}{t_+ + t'_+}\frac{t_- - t'_-}{t_- + t'_-}e^{2m(\eta'_- - \eta'_+)}, \tag{81b}$$

$$C \sim \frac{t_- - t'_-}{t_- + t'_-}e^{2m\eta'_-}, \tag{81c}$$

where $t_+ = t(z_+)$, $\eta'_- = \eta(z'_-)$, $z'_- = \kappa' a_-/m$, etc. If we now insert this approximation into the form for the energy density, we find

$$\begin{aligned} u = \langle T^{00}\rangle = \frac{1}{8\pi^2 a_+^4}2\sum_{m=1}^\infty m\int_0^\infty dz_+\, z_+ t'_r \\ \times\Bigg\{\left[\frac{t_+ - t'_+}{t_+ + t'_+}e^{2m(\eta'_r - \eta'_+)} + \frac{t_- - t'_-}{t_- + t'_-}e^{2m(-\eta'_r + \eta'_-)}\right] \\ \times\left[\frac{m^2 z_+^2}{2}(1-8\xi) + \left(\frac{\lambda h a_+^2}{a} + \frac{m^2 a_+^2}{r^2}\right)(1-4\xi)\right] \\ - m^2 z_+^2\frac{t_+ - t'_+}{t_+ + t'_+}\frac{t_- - t'_-}{t_- + t'_-}e^{2m(\eta'_- - \eta'_+)}\Bigg\}. \end{aligned} \tag{82}$$

If we are interested in the surface divergence as r approaches the outer radius a_+ from within the annulus, the dominant term comes from the first exponential factor only. Because we are considering the limit $\lambda h a \ll m^2$, we have

$$t'_+ \approx t_+\left(1 - \frac{\lambda h}{2m^2}\frac{a_+^2}{a}t_+^2\right), \tag{83}$$

and we have

$$u \sim -\frac{\lambda h/a}{32\pi^2 a_+^2}\sum_{m=1}^\infty m\int_0^\infty dz\, zt(1 - 8\xi + t^2)e^{2m(\eta'_r - \eta'_+)}. \tag{84}$$

The sum over m is carried out according to Eq. (71), or

$$\sum_{m=1}^{\infty} m e^{2m(\eta_r'-\eta_+')} \sim \left(\frac{r t_r'}{2(r-a_+)}\right)^2, \tag{85}$$

and the remaining integrals over z are elementary. The result is

$$u \sim \frac{\lambda h}{96\pi^2 a}\frac{1-6\xi}{(r-a_+)^2}, \quad r \to a_+, \tag{86}$$

the expected universal divergence of a scalar field near a surface of discontinuity,[30] without significance, which may be eliminated by setting $\xi = 1/6$.

7.3. *Surface energy*

Now we want to establish that the surface energy $\mathfrak{E}$ (60) is the same as the integrated local energy density in the annulus when the limit $\delta \to 0$ is taken. To examine this limit, we consider $\lambda h/a \gg \kappa^2$. So we apply the uniform asymptotic expansion for the Bessel functions of κ' only. We must keep the first two terms in powers of $\kappa \ll \kappa'$:

$$\begin{aligned}\Xi \sim& -\kappa'^2 \frac{I_m(\kappa a_-)K_m(\kappa a_+)}{m z_-' z_+' \sqrt{t_-' t_+'}} \sinh m(\eta_-' - \eta_+') \\ &-\frac{\kappa'\kappa}{m}\left[\frac{1}{z_+'}\sqrt{\frac{t_-'}{t_+'}} I_m'(\kappa a_-)K_m(\kappa a_+) - \frac{1}{z_-'}\sqrt{\frac{t_+'}{t_-'}} I_m(\kappa a_-)K_m'(\kappa a_+)\right] \\ &\quad\times \cosh m(\eta_-' - \eta_+').\end{aligned} \tag{87}$$

Because we are now regarding the shell as very thin,

$$\eta_-' - \eta_+' \approx -\frac{\delta}{a}\frac{1}{t'}, \tag{88}$$

where

$$t' \sim \frac{1}{z'} \sim \frac{m}{\sqrt{\lambda h a}}, \tag{89}$$

using the Wronskian (7) we get the denominator

$$\Xi \sim -\frac{1}{a^2}[1 + \lambda I_m(\kappa a)K_m(\kappa a)]. \tag{90}$$

Then we immediately find the interior coefficients:

$$A \sim \frac{\pi}{2}\sqrt{\lambda h a}\frac{I_m(\kappa a)K_m(\kappa a)}{1+\lambda I_m(\kappa a)K_m(\kappa a)} e^{-2m\eta'}, \tag{91a}$$

$$B \sim \frac{1}{2}\sqrt{\lambda h a}\frac{I_m(\kappa a)K_m(\kappa a)}{1+\lambda I_m(\kappa a)K_m(\kappa a)}, \tag{91b}$$

$$C \sim \frac{1}{2\pi}\sqrt{\lambda h a}\frac{I_m(\kappa a)K_m(\kappa a)}{1+\lambda I_m(\kappa a)K_m(\kappa a)} e^{2m\eta'}. \tag{91c}$$

7.4. *Identity of shell energy and surface energy*

We now insert this in the expression for the energy density (80) and keep only the largest terms, thereby neglecting κ^2 relative to $\lambda h/a$. This gives a leading term proportional to h, which when multiplied by the area of the annulus $2\pi a\delta$ gives for the energy in the shell

$$\mathcal{E}_{\rm ann} \sim (1-4\xi)\frac{\lambda}{4\pi a^2}\sum_{m=-\infty}^{\infty}\int_0^\infty d\kappa a\,\kappa a\frac{I_m(\kappa a)K_m(\kappa a)}{1+\lambda I_m(\kappa a)K_m(\kappa a)}, \tag{92}$$

which is exactly the form of the surface energy $\mathfrak{E}$ given by the negative of the second term in the integrated energy density (59).

7.5. *Renormalizability of surface energy*

In particular, note that the term in $\mathfrak{E}$ of order λ^3 is, for the conformal value $\xi = 1/6$, exactly equal to that term in the total energy $\mathcal{E}$ in Eq. (46):

$$\mathfrak{E}^{(3)} = \mathcal{E}^{(3)}. \tag{93}$$

This means that the divergence encountered in the global energy is exactly accounted for by the divergence in the surface energy, which would seem to provide strong evidence in favor of the renormalizablity of that divergence.

8. Conclusion

The work reported here and in Refs. 20,29 represents a significant advance in understanding the divergence structure of Casimir self-energies. We have shown that the surface energy of a δ-function shell potential is in fact the integrated local energy density contained within the shell when the shell is given a finite thickness. That surface energy contains the entire third-order divergence in the total Casimir energy. The local Casimir energy diverges as the shell is approached, but that divergence is integrable, so it yields a finite contribution to the total energy. The identification of the divergent part of the total energy with that associated with the surface strongly suggests that this divergence can be absorbed in a renormalization of parameters describing the background potential.

Challenges yet remain. This renormalization procedure needs to be made precise. Further, we must make more progress in understanding the sign (and for cylindrical geometries, the vanishing) of the total Casimir self-energy. And, of course, we must understand the implications of surface divergences on the coupling to gravity. Work is proceeding in all these directions.

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