

International Journal of Modern Physics A
 © World Scientific Publishing Company

Recent Developments in Quantum Vacuum Energy for Confined Fields

K. A. MILTON

Oklahoma Center for High Energy Physics, University of Oklahoma, Norman, OK 73072 USA

Quantum vacuum energy entered hadronic physics through the zero-point energy parameter introduced into the bag model. Estimates of this parameter led to apparent discordance with phenomenological fits. More serious were divergences which were omitted in an *ad hoc* manner. New developments in understanding Casimir self-stresses, and the nature of surface divergences, promise to render the situation clearcut.

Keywords: Casimir energy; divergences; surface terms.

1. Introduction

In the bag model,¹ the normal vacuum is a perfect color magnetic conductor, that is, the color magnetic permeability μ is infinite, while the vacuum in the interior of the bag is characterized by $\mu = 1$. This implies that the color electric and magnetic fields are confined to the interior of the bag, and that they satisfy the following boundary conditions on its surface S : $\mathbf{n} \cdot \mathbf{E}|_S = 0$, $\mathbf{n} \times \mathbf{B}|_S = 0$, where \mathbf{n} is a unit normal to S . Now, even in an “empty” bag (i.e., one containing no quarks) there will be nonzero fields present because of quantum fluctuations. This gives rise to a zero-point or Casimir energy.

The bag-model Lagrangian is

$$\mathcal{L}_{\text{bag}} = (\mathcal{L}_D + \mathcal{L}_{\text{YM}} - B)\eta(\bar{\psi}\psi), \quad (1)$$

where $\mathcal{L}_{D,\text{YM}}$ are the Dirac and Yang-Mills Lagrangians, respectively, B is the bag constant, and η is the unit step function. Variation of \mathcal{L}_{bag} leads to the following linear and quadratic boundary conditions (n^μ is the outwardly directed normal) $-i\gamma n\psi = \psi$ and $n_\alpha G^{\alpha\beta} = 0$, and $-\frac{1}{4}G^2 - \frac{1}{2}(n\partial)\bar{\psi}\psi = B$, in terms of its quark contribution.

The MIT bag model consists in solving the Dirac equation subject to the boundary conditions, which in the simplest realization refer to a spherical cavity approximation. The interactions, through the gauge fields, are treated perturbatively in the approximation that the strong coupling is regarded as small, $\alpha_s \ll 1$. The radial wavefunctions are then expressed in terms of spherical Bessel functions. The results of fits, shown in Table 1 are essentially identical with those found by DeGrand et al.,¹ Model E. The following comments can be offered concerning these fits.

2 *K. A. Milton*

Table 1. Fits to hadron masses for various bag models.

Parameters and Masses	Experimental Values	Model A	Model D	Model E
Z		-1.0	1.0	1.8327
$B^{1/4}(\text{GeV})$		0.19526	0.16515	0.14535
$F (\text{GeV}^2)$		-0.21923	-0.064729	0
α_s		2.16066	2.17932	2.19897
$m_s (\text{GeV})$		0.2807	0.2802	0.2797
$R_N (\text{GeV}^{-1})$		5.06	5.03	5.01
$\delta m (\text{GeV})$		0.1325	0.1343	0.1408
m_π	0.138	0.3610	0.3229	0.2833
m_η	0.549	0.645	0.6105	0.5583
$m_{\eta'}$	0.958	0.5031	0.4702	0.4209
m_K	0.4957	0.5827	0.5489	0.4982
m_N	<u>0.9389</u>	<u>0.9389</u>	<u>0.9389</u>	<u>0.9389</u>
m_Λ	1.1156	1.1022	1.1017	1.1012
m_Σ	1.1931	1.1426	1.1426	1.1422
m_Ξ	1.13181	1.2885	1.2876	1.1867
m_ρ	0.769	0.7826	0.7826	0.7826
m_ω	<u>0.7826</u>	<u>0.7826</u>	<u>0.7826</u>	<u>0.7826</u>
m_ϕ	1.0195	1.0733	1.0711	1.0688
m_{K^*}	0.8921	0.9251	0.9246	0.9229
m_Δ	<u>1.232</u>	<u>1.232</u>	<u>1.232</u>	<u>1.232</u>
m_{Σ^*}	1.3839	1.3760	1.3761	1.3762
m_{Ξ^*}	1.5334	1.5229	1.5229	1.5230
m_Ω	<u>1.6725</u>	<u>1.6725</u>	<u>1.6725</u>	<u>1.6725</u>

- The $Z = -1$ (repulsive) estimate (my best guess in 1983²), model A, agrees about as well with the data as does the optimal bag model E.
- All the models agree more closely with each other than with the data.
- Chiral bag models can be used to improve the fits with the pseudoscalars.
- For models A and D a constant force term is included: (the bag radius is a)

$$H = BV + \sigma A + Fa + \frac{Z}{a} + H_{q,g}. \quad (2)$$

- Some details of how Z is calculated will follow.

1.1. *Gluon and quark condensates*

Due to zero-point fluctuations, confinement will result in a nonzero expectation value for the squared field strength, ($\delta \rightarrow 0$ is a time-splitting regulator)

$$\langle G^2(r) \rangle = \frac{1}{4\pi^2 a} \frac{1}{r^2} \frac{d}{dr} \sum_{l=1}^{\infty} (2l+1) 2 \int_0^{\infty} dx e^{ix\delta} \frac{s_l(xr/a) s'_l(xr/a)}{s_l(x) s'_l(x)}, \quad (3)$$

Similarly, we obtain the following expression for the quark condensate

$$\langle \bar{q}q(r) \rangle = -\frac{1}{4\pi^2 ar^2} \sum_{l=0}^{\infty} 2(l+1) \int_{-\infty}^{\infty} dx e^{ix\delta} \frac{s_l^2(xr/a) + s_{l+1}^2(xr/a)}{s_l^2(x) + s_{l+1}^2(x)}. \quad (4)$$

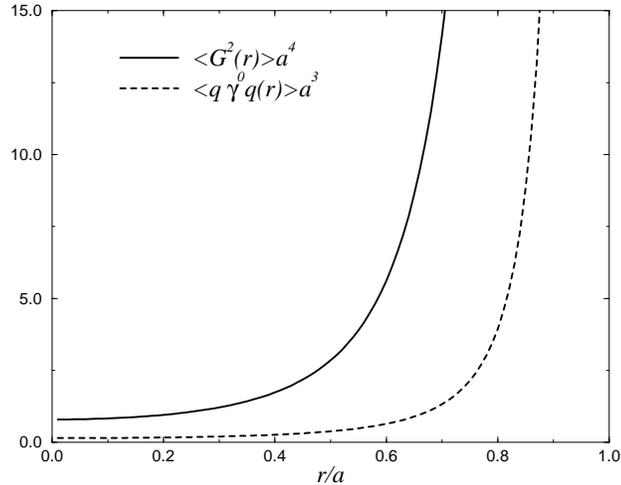


Fig. 1. Magnitude of the quark and gluon condensates as functions of the distance from the center of the bag. The values shown are for a single color, and, for the quarks, for a single flavor and helicity state.

Here s_l and e_l are modified spherical Bessel functions defined in (10) below. Details of these calculations were given in Ref. 3. Graphs of these condensate densities are shown in Fig. 1. The central values seem to agree with the observed values of the quark and gluon condensates,⁴ but they diverge as the surface of the bag is approached.

2. Casimir Forces on Spheres

The calculations presented here were carried out in response to the recent program of the MIT group.⁵ They rediscovered irremovable divergences in the Casimir energy for a circle first discovered by Sen in 1981,⁶ but then found divergences in the case of a spherical surface, thereby casting doubt on the validity of the Boyer calculation for the Casimir energy of a perfectly conducting spherical shell.⁷ Some of their results, as we shall see, are spurious, and the rest had been earlier discovered by the Leipzig group.⁸ However, the MIT group's work has been valuable in sparking new investigations of the problems of surface energies and divergences.⁹

2.1. δ -function Potential

We now carry out a calculation¹⁰ of the zero-point energy in three spatial dimensions, with a radially symmetric singular background

$$\mathcal{L}_{\text{int}} = -\frac{1}{2} \frac{\lambda}{a} \delta(r-a) \phi^2(x), \quad (5)$$

4 *K. A. Milton*

which would correspond to a Dirichlet shell in the limit $\lambda \rightarrow \infty$. The time-Fourier transformed Green's function satisfies the equation ($\kappa^2 = -\omega^2$)

$$\left[-\nabla^2 + \kappa^2 + \frac{\lambda}{a}\delta(r-a)\right]G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \quad (6)$$

We write G in terms of a reduced Green's function

$$G(\mathbf{r}, \mathbf{r}') = \sum_{lm} g_l(r, r') Y_{lm}(\Omega) Y_{lm}^*(\Omega'), \quad (7)$$

where g_l satisfies

$$\left[-\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} + \frac{l(l+1)}{r^2} + \kappa^2 + \frac{\lambda}{a}\delta(r-a)\right] g_l(r, r') = \frac{1}{r^2} \delta(r - r'). \quad (8)$$

The solution inside the sphere ($0 < r, r' < a$) is

$$g_l(r, r') = \frac{1}{\kappa r r'} \left[e_l(\kappa r_>) s_l(\kappa r_<) - \frac{\lambda}{\kappa a} e_l^2(\kappa a) \frac{s_l(\kappa r) s_l(\kappa r')}{1 + \frac{\lambda}{\kappa a} s_l(\kappa a) e_l(\kappa a)} \right]. \quad (9)$$

Here we have introduced the modified Riccati-Bessel functions,

$$s_l(x) = \sqrt{\frac{\pi x}{2}} I_{l+1/2}(x), \quad e_l(x) = \sqrt{\frac{2x}{\pi}} K_{l+1/2}(x). \quad (10)$$

Note that this reduces to the expected result, vanishing as $r \rightarrow a$, in the limit of strong coupling. When both points are outside the sphere, $r, r' > a$, we obtain a similar result, with $e_l \leftrightarrow s_l$ in the second term.

2.2. Pressure on sphere

Now we want to get the radial-radial component of the stress tensor to extract the pressure on the sphere, which is obtained by applying the operator

$$\partial_r \partial_{r'} - \frac{1}{2} (-\partial^0 \partial'^0 + \nabla \cdot \nabla') \rightarrow \frac{1}{2} \left[\partial_r \partial_{r'} - \kappa^2 - \frac{l(l+1)}{r^2} \right] \quad (11)$$

to the Green's function, where in the last term we have averaged over the surface of the sphere. In this way we find, from the discontinuity of $\langle T_{rr} \rangle$ across the $r = a$ surface, the net stress, which can alternatively be obtained by differentiating with respect to a , with λ/a fixed, the total Casimir energy

$$E = -\frac{1}{2\pi a} \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} dx x \frac{d}{dx} \ln [1 + \lambda I_\nu(x) K_\nu(x)]. \quad (12)$$

This expression, upon integration by parts, coincides with that given recently by Barton,¹¹ and was first analyzed in detail by Scandurra.¹²

2.3. Strong coupling

For strong coupling, the above energy reduces to the well-known expression for the Casimir energy of a massless scalar field inside and outside a sphere upon which Dirichlet boundary conditions are imposed, that is, that the field must vanish at $r = a$: ($\nu = l + 1/2$)

$$\lim_{\lambda \rightarrow \infty} E = -\frac{1}{2\pi a} \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} dx x \frac{d}{dx} \ln [I_{\nu}(x)K_{\nu}(x)], \quad (13)$$

because multiplying the argument of the logarithm by a power of x is without effect, corresponding to a contact term. This may be evaluated numerically:¹³ $E^{\text{TE}} = 0.002817/a$, which is much smaller than the Boyer result for electrodynamics:⁷ $E^{\text{EM}} = 0.04618/a$, although both are repulsive.

2.4. Weak coupling

The opposite limit is of interest here. The expansion of the logarithm in (12) is immediate for small λ . The first term, of order λ , is evidently divergent, but irrelevant, since that may be removed by renormalization of the tadpole graph. In contradistinction to the claim of the MIT group⁵ the order λ^2 term is finite. That term is

$$E^{(\lambda^2)} = \frac{\lambda^2}{4\pi a} \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} dx x \frac{d}{dx} [I_{l+1/2}(x)K_{l+1/2}(x)]^2. \quad (14)$$

The sum on l can be carried out using a trick due to Klich:¹⁴ The sum rule

$$\sum_{l=0}^{\infty} (2l+1) e_l(x) s_l(y) P_l(\cos \theta) = \frac{xy}{\rho} e^{-\rho}, \quad (15)$$

where $\rho = \sqrt{x^2 + y^2 - 2xy \cos \theta}$, is squared, and then integrated over θ . In this way we learn that

$$\sum_{l=0}^{\infty} (2l+1) e_l^2(x) s_l^2(x) = \frac{x^2}{2} \int_0^{4x} \frac{dw}{w} e^{-w}. \quad (16)$$

Although this integral is divergent, because we did not integrate (12) by parts, that divergence does not contribute:

$$E^{(\lambda^2)} = \frac{\lambda^2}{4\pi a} \int_0^{\infty} dx \frac{1}{2} x \frac{d}{dx} \int_0^{4x} \frac{dw}{w} e^{-w} = \frac{\lambda^2}{32\pi a}, \quad (17)$$

which is exactly the result I had found earlier,¹⁵ based on the following formula for a hypersphere in D dimensions:

$$E_D^{(\lambda^2)} = -\frac{\lambda^2}{\pi a} \frac{\Gamma(\frac{D-1}{2}) \Gamma(D-3/2) \Gamma(1-D/2)}{2^{1+2D} [\Gamma(D/2)]^2}. \quad (18)$$

which exhibits poles when D is even, where the Casimir energy is known to diverge.¹³

6 *K. A. Milton*

2.5. Divergences

However, before we wax too euphoric, we recognize that the order λ^3 term appears logarithmically divergent, just as the MIT group claimed.⁵ To study the behavior of the sum for large values of l , we can use the uniform asymptotic expansion (Debye expansion),

$$\nu \gg 1: \quad I_\nu(x)K_\nu(x) \sim \frac{t}{2\nu} \left[1 + \frac{A(t)}{\nu^2} + \frac{B(t)}{\nu^4} + \dots \right]. \quad (19)$$

Here $x = \nu z$, and $t = 1/\sqrt{1+z^2}$. The functions A and B , etc., are polynomials in t . We now insert this into the energy expression and expand not in λ but in ν ; the leading term is

$$E(\lambda^3) \sim \frac{\lambda^3}{24\pi a} \sum_{l=0}^{\infty} \frac{1}{\nu} \int_0^{\infty} \frac{dz}{(1+z^2)^{3/2}} = \frac{\lambda^3}{24\pi a} \zeta(1). \quad (20)$$

Although the frequency integral is finite, the angular momentum sum is divergent. The appearance here of the divergent $\zeta(1)$ seems to signal an insuperable barrier to extraction of a finite Casimir energy for finite λ . The situation is different in the limit $\lambda \rightarrow \infty$.

This divergence has been known for many years, and was first calculated explicitly in 1998 by Bordag et al.,⁸ where the second heat kernel coefficient gave

$$E \sim \frac{\lambda^3}{48\pi a} \frac{1}{s}, \quad s \rightarrow 0. \quad (21)$$

A possible way of dealing with this divergence was advocated in Scandurra.¹² Very recently, Bordag and Vassilevich¹⁶ have reanalyzed such problems from the heat kernel approach. They show that this $O(\lambda^3)$ divergence corresponds to a surface tension counterterm, an idea proposed by me¹⁷ in 1980 in connection with the zero-point energy contribution to the bag model. Such a surface term corresponds to λ/a fixed, which then necessarily implies a divergence of order λ^3 . Bordag and Vassilevich¹⁶ argue that it is perfectly appropriate to render this divergence finite by renormalization.

2.6. Boundary layer energy

Here we show¹⁰ that the surface energy can be interpreted as the bulk energy of the boundary layer. We do this by considering a scalar field in $1+1+d$ dimensions interacting with the background $\mathcal{L}_{\text{int}} = -\frac{\lambda}{2}\phi^2\sigma$, where

$$\sigma(x) = \begin{cases} h, & -\frac{\delta}{2} < x < \frac{\delta}{2}, \\ 0, & \text{otherwise,} \end{cases} \quad (22)$$

with the property that $h\delta = 1$. The reduced Green's function satisfies ($\kappa^2 = k_\perp^2 - \omega^2$)

$$\left[-\frac{\partial^2}{\partial x^2} + \kappa^2 + \lambda\sigma(x) \right] g(x, x') = \delta(x - x'). \quad (23)$$

This may be easily solved in the region of the slab, $-\frac{\delta}{2} < x < \frac{\delta}{2}$, ($\kappa' = \sqrt{\kappa^2 + \lambda h}$)

$$g(x, x') = \frac{1}{2\kappa'} \left\{ e^{-\kappa'|x-x'|} + \frac{1}{\hat{\Delta}} \left[(\kappa'^2 - \kappa^2) \cosh \kappa'(x+x') + (\kappa' - \kappa)^2 e^{-\kappa'\delta} \cosh \kappa'(x-x') \right] \right\}, \quad (24)$$

$$\hat{\Delta} = 2\kappa\kappa' \cosh \kappa'\delta + (\kappa^2 + \kappa'^2) \sinh \kappa'\delta. \quad (25)$$

We generalize the considerations of Graham and Olum,¹⁸ by considering the stress tensor with an arbitrary conformal term,

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} g^{\mu\nu} (\partial_\lambda \phi \partial^\lambda \phi + \lambda h \phi^2) - \alpha (\partial^\mu \partial^\nu - g^{\mu\nu} \partial^2) \phi^2. \quad (26)$$

We get the following form for the energy density within the slab,

$$T^{00} = \frac{2^{-d-2} \pi^{-(d+1)/2}}{\Gamma((d+3)/2)} \int_0^\infty \frac{d\kappa \kappa^d}{\kappa' \hat{\Delta}} \left\{ (\kappa'^2 - \kappa^2) [(1-4\alpha)(1+d)\kappa'^2 - \kappa^2] \cosh 2\kappa'x - (\kappa' - \kappa)^2 e^{-\kappa'\delta} \kappa^2 \right\}. \quad (27)$$

From this we can calculate the behavior of the energy density as the boundary is approached from the inside:

$$T^{00} \sim \frac{\Gamma(d+1)\lambda h}{2^{d+4} \pi^{(d+1)/2} \Gamma((d+3)/2)} \frac{1-4\alpha(d+1)/d}{(\delta-2|x|)^d}, \quad |x| \rightarrow \delta/2. \quad (28)$$

For $d=2$ for example, this agrees with the result found by Graham and Olum for $\alpha=0$:

$$T^{00} \sim \frac{\lambda h}{96\pi^2} \frac{(1-6\alpha)}{(\delta/2-|x|)^d}, \quad |x| \rightarrow \frac{\delta}{2}. \quad (29)$$

Note that, as we expect, this surface divergence vanishes for the conformal stress tensor, where $\alpha = d/4(d+1)$. (There will be subleading divergences if $d > 2$.)

We can also calculate the energy density on the other side of the boundary, from the Green's function for $x, x' < -\delta/2$,

$$g(x, x') = \frac{1}{2\kappa} \left[e^{-\kappa|x-x'|} - e^{\kappa(x+x'+\delta)} (\kappa'^2 - \kappa^2) \frac{\sinh \kappa'\delta}{\hat{\Delta}} \right], \quad (30)$$

and the corresponding energy density is given by

$$T^{00} = -\frac{d(1-4\alpha(d+1)/d)}{2^{d+2} \pi^{(d+1)/2} \Gamma((d+3)/2)} \int_0^\infty d\kappa \kappa^{d+1} \frac{(\kappa'^2 - \kappa^2)}{\hat{\Delta}} e^{2\kappa(x+\delta/2)} \sinh \kappa'\delta, \quad (31)$$

which vanishes if the conformal value of α is used. The divergent term, as $x \rightarrow -\delta/2$, is just the negative of that found on the inside. This is why, when the total energy is computed by integrating the energy density, it is finite for $d < 2$, and independent of α . The divergence encountered for $d=2$ may be handled by renormalization of

8 *K. A. Milton*

the interaction potential. In the limit as $h \rightarrow \infty$, $h\delta = 1$, we recover the divergent expression for a single interface

$$\lim_{h \rightarrow \infty} E_s = \frac{1}{2^{d+2}\pi^{(d+1)/2}\Gamma((d+3)/2)} \int_0^\infty d\kappa \kappa^d \frac{\lambda}{\lambda + 2\kappa}. \quad (32)$$

Therefore, surface divergences have an illusory character.

2.7. Dielectric Spheres

The Casimir self-stress on a uniform dielectric sphere was first worked out by me in 1979.¹⁹ It was generalized to the case when both electric permittivity and magnetic permeability are present in 1997.²⁰ The result for the pressure ($x = \sqrt{\varepsilon\mu}|y|$, $x' = \sqrt{\varepsilon'\mu'}|y|$ where ε', μ' are the interior, and ε, μ are the exterior, values of the permittivity and the permeability) is

$$P = -\frac{1}{2a^4} \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{iy\delta} \sum_{l=1}^{\infty} \frac{2l+1}{4\pi} \left\{ x \frac{d}{dx} \ln D_l + 2x' [s'_l(x')e'_l(x') - e_l(x')s''_l(x')] - 2x [s'_l(x)e'_l(x) - e_l(x)s''_l(x)] \right\}, \quad (33)$$

where the ‘‘bulk’’ pressure has been subtracted, and

$$D_l = [s_l(x')e'_l(x) - s'_l(x')e_l(x)]^2 - \xi^2 [s_l(x')e'_l(x) + s'_l(x')e_l(x)]^2, \quad (34)$$

with the parameter ξ being

$$\xi = \frac{\sqrt{\frac{\varepsilon'}{\varepsilon} \frac{\mu}{\mu'}} - 1}{\sqrt{\frac{\varepsilon'}{\varepsilon} \frac{\mu}{\mu'}} + 1}, \quad (35)$$

and $\delta \rightarrow 0$ is the temporal regulator. This result is obtained either by computing the radial-radial component of the stress tensor, or from the total energy.

In general, this result is divergent. However, consider the special case $\sqrt{\varepsilon\mu} = \sqrt{\varepsilon'\mu'}$, that is, when the speed of light is the same in both media. Then $x = x'$ and the Casimir energy reduces to

$$E = -\frac{1}{4\pi a} \int_{-\infty}^{\infty} dy e^{iy\delta} \sum_{l=1}^{\infty} (2l+1)x \frac{d}{dx} \ln[1 - \xi^2 ((s_l e_l)')^2], \quad (36)$$

where

$$\xi = \frac{\mu - \mu'}{\mu + \mu'} = -\frac{\varepsilon - \varepsilon'}{\varepsilon + \varepsilon'}. \quad (37)$$

If $\xi = 1$ we recover the case of a perfectly conducting spherical shell, treated above. In fact E is finite for all ξ .

Of particular interest is the dilute limit, where¹⁴

$$E \approx \frac{5\xi^2}{32\pi a} = \frac{0.0994718\xi^2}{2a}, \quad \xi \ll 1. \quad (38)$$

It is remarkable that the value for a spherical conducting shell, for which $\xi = 1$, is only 7% lower, which as Klich remarks,¹⁴ is accounted for nearly entirely by the next term in the small ξ expansion.

There is another dilute limit which is also quite surprising. For a purely dielectric sphere ($\mu = 1$) the leading term in an expansion in powers of $\varepsilon - 1$ is finite:^{21,8}

$$E = \frac{23}{1536\pi} \frac{(\varepsilon - 1)^2}{a} = (\varepsilon - 1)^2 \frac{0.004767}{a}. \quad (39)$$

This result coincides with the sum of van der Waals energies of the material making up the ball as Ng and I showed earlier in 1998.²² The term of order $(\varepsilon - 1)^3$ is divergent.⁸ The establishment of this result was the death knell for the Casimir energy explanation of sonoluminescence.²³

3. Dielectric Cylinders

The fundamental difficulty in cylindrical geometries is that there is in general no decoupling between TE and TM modes. Progress in understanding has therefore been much slower in this regime. It was only in 1981 that it was found that the electromagnetic Casimir energy of a perfectly conducting cylinder was attractive, the energy per unit length being²⁴ $\mathcal{E}_{\text{em,cyl}} = -0.01356/a^2$, for a circular cylinder of radius a . The corresponding result for a scalar field satisfying Dirichlet boundary conditions of the cylinder is repulsive,²⁵ $\mathcal{E}_{\text{D,cyl}} = 0.000606/a^2$. These ideal limits are finite, but, as with the spherical geometry, less ideal configurations have unremovable divergences. For example, a cylindrical δ -shell potential has divergences (in third order).²⁶ And it is expected that a dielectric cylinder will have a divergent Casimir energy, although the coefficient of $(\varepsilon - 1)^2$ will be finite for a dilute dielectric cylinder,²⁷ corresponding to a finite van der Waals energy between the molecules that make up the material. In fact, a calculation of the renormalized van der Waals energy for a dilute dielectric cylinder gives zero, as is the Casimir energy for a cylinder for which the speed of light is the same inside and out, because $\varepsilon\mu = 1$.²⁸ A calculation of the Casimir energy for a dielectric cylinder is in progress with my graduate student Ines Cavero-Pelaez.

4. Conclusions

We began by reviewing ancient results for the contributions of zero-point energies in hadronic (bag) models. We saw that in general there are surface divergences in the condensates, and in the energy densities, for the quark and gluon fields. We discussed δ -function potentials, which in general give divergent results in 3rd order, but are finite in strong coupling. These divergences are largely identified with surface energies, which can be interpreted as bulk energies when the boundaries are smoothed. Precisely analogous phenomena happen for dielectric balls and cylinders (although there is some remarkable symmetry buried in the latter). I hope this improved understanding of divergences can lead to improved hadronic models. For

10 *K. A. Milton*

the current status of Casimir phenomena, see my review article,²⁹ where complete references can be found.

Acknowledgments

This work was supported in part by the US Department of Energy. I thank H. Fried for inviting me to participate in this fascinating meeting.

References

1. A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. Weisskopf, *Phys. Rev. D* **9**, 3471 (1974); T. DeGrand, R. L. Jaffe, K. Johnson, and J. Kiskis, *Phys. Rev. D* **12**, 2060 (1975).
2. K. A. Milton, *Phys. Rev. D* **27**, 439 (1983).
3. K. A. Milton, *Phys. Lett.* **104B**, 49 (1981).
4. A. I. Vainshtein, V. I. Zakharov, and M. A. Shifman, *Nucl. Phys. B* **147**, 385, 448, 519 (1979).
5. N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, O. Schroeder, and H. Weigel, *Nucl. Phys. B* **677**, 379 (2004), and references therein.
6. S. Sen, *Phys. Rev. D* **24**, 869 (1981); *J. Math. Phys.* **22**, 2968 (1981).
7. T. H. Boyer, *Phys. Rev.* **174**, 1764 (1968).
8. M. Bordag, K. Kirsten, and D. Vassilevich, *Phys. Rev. D* **59**, 085011 (1999).
9. S. A. Fulling, *J. Phys. A* **36**, 6529 (2003).
10. K. A. Milton, *J. Phys. A* **37**, 6391 (2004).
11. G. Barton, *J. Phys. A* **37**, 1011 (2004).
12. M. Scandurra, *J. Phys. A* **32**, 5679 (1999).
13. C. M. Bender and K. A. Milton, *Phys. Rev. D* **50**, 6547 (1994).
14. I. Klich, *Phys. Rev. D* **61**, 025004 (2000).
15. K. A. Milton, *Phys. Rev. D* **68**, 065020 (2003).
16. M. Bordag and D. V. Vassilevich, *Phys. Rev. D* **70**, 045003 (2004).
17. K. A. Milton, *Phys. Rev. D* **22**, 1441, 1444 (1980).
18. N. Graham and K. D. Olum, *Phys. Rev. D* **67**, 085014 (2003); K. D. Olum and N. Graham, *Phys. Lett.* **B554**, 175 (2003).
19. K. A. Milton, *Ann. Phys. (N.Y.)* **127**, 49 (1980).
20. K. A. Milton and Y. J. Ng, *Phys. Rev. E* **55**, 4207 (1997).
21. I. Brevik, V. N. Marachevsky, and K. A. Milton, *Phys. Rev. Lett.* **82**, 3948 (1999); G. Barton, *J. Phys. A* **32**, 525 (1999); J. S. Høye and I. Brevik, *J. Stat. Phys.* **100**, 223 (2000).
22. K. A. Milton and J. Y. Ng, *Phys. Rev. E* **57**, 5504 (1998).
23. M. P. Brenner, S. Hilgenfeldt, and D. Lohse, *Rev. Mod. Phys.* **74**, 425 (2002).
24. L. L. DeRaad, Jr., and K. A. Milton, *Ann. Phys. (N.Y.)* **136**, 229 (1981).
25. V. V. Nesterenko and I. G. Pirozhenko, *J. Math. Phys.* **41**, 4521 (2000).
26. M. Scandurra, *J. Phys. A*, **33**, 5707 (2000).
27. M. Bordag and I. G. Pirozhenko, *Phys. Rev. D*, **64**, 025019 (2001).
28. K. A. Milton, A. V. Nesterenko, and V. V. Nesterenko, *Phys. Rev. D* **59**, 105009 (1999).
29. K. A. Milton, *J. Phys. A* **37**, R209 (2004).