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# Constraints on Extra Dimensions From Cosmological and Terrestrial Measurements

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If quantum fields exist in extra compact dimensions, they will give rise to a quantum vacuum or Casimir energy. That vacuum energy will manifest itself as a cosmological constant. The fact that supernova and cosmic microwave background data indicate that the cosmological constant is of the same order as the critical mass density to close the universe supplies a lower bound on the size of the extra dimensions. Recent laboratory constraints on deviations from Newton's law place an upper limit. The allowed region is so small as to suggest that either extra compact dimensions do not exist, or their properties are about to be tightly constrained by experimental data.

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## 1. Introduction

It has been appreciated for many years that there is an apparently fundamental conflict between quantum field theory and the smallness of the cosmological constant [1]. This is because the zero-point energy of the quantum fields (including gravity) in the universe should give rise to an observable cosmological vacuum energy density,

$$u_{\rm cosmo} \sim \frac{1}{L_{\rm Pl}^4}, \tag{1}$$

where the Planck length is

$$L_{\rm Pl} = \sqrt{G_N} = 1.6 \times 10^{-33}\,{\rm cm}. \tag{2}$$

(We use natural units with $\hbar = c = 1$. The conversion factor is $\hbar c \simeq 2\times 10^{-14}\,{\rm GeV\,cm}$.) This means that the cosmic vacuum energy density would be

$$u_{\rm cosmo} \sim 10^{118}\ {\rm GeV\,cm}^{-3}, \tag{3}$$

which is 123 orders of magnitude larger than the critical mass density required to close the universe:

$$\rho_c = \frac{3H_0^2}{8\pi G_N} = 1.05\times 10^{-5} h_0^2\,{\rm GeV\,cm}^{-3}, \tag{4}$$

in terms of the dimensionless Hubble constant, $h_0 = H_0/100\ \ {\rm km\,s^{-1}Mpc^{-1}}$. From relativistic covariance the cosmological vacuum energy density must be the 00 component of the expectation value of the energy-momentum tensor, which we can identify with the cosmological constant:

$$\langle T^{\mu\nu}\rangle = -u g^{\mu\nu} = -\frac{\Lambda}{8\pi G} g^{\mu\nu}. \tag{5}$$

[We use the metric with signature $(-1,1,1,1)$.] Of course this is absurd with $u$ given by Eq. (3), which would have caused the universe to expand to zero density long ago.

For most of the past century, it was the prejudice of theoreticians that the cosmological constant was exactly zero, although no one could give a convincing argument. Recently, however, with the new data gathered on the brightness-redshift relation for very distant type Ia supernovæ [2, 3], corroborated by the balloon observations of the anisotropy in the cosmic microwave background [4, 5, 6], it seems clear that the cosmological constant is near the critical value, or $\Omega_\Lambda = \Lambda/8\pi G\rho_c \sim 1$. It is very hard to understand how the cosmological constant can be nonzero but small.

We here present a plausible scenario for understanding this puzzle. It is reasonable (but by no means established) that vacuum fluctuations in the gravitational and matter fields in flat Minkowski space give a zero cosmological constant.[1] (See below.) Effects due to curvature are negligible. But since the work of Kaluza and Klein [8] it has been an exciting possibility that there exist extra dimensions beyond those of Minkowski space-time. Why do we not experience those dimensions? The simplest possibility seems to be that those extra dimensions are curled up in a space $\mathcal{S}$ of size $a$, smaller than some observable limit.

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[1] This is in line with considerations of Casimir energies in other contexts. For example, although the electromagnetic Casimir energy of a ball of dilute nondispersive dielectric material is divergent, that divergence can be unambiguously removed as an unobservable bulk and surface effect, and a unique finite energy, interpretable as a sum of van der Waals energies, emerges. See Refs. [7].

Of course, in recent years, the idea of extra dimensions has become much more compelling. Superstring theory requires at least 10 dimensions, six of which must be compactified, and the putative M theory, supergravity, is an 11 dimensional theory. Perhaps, if only gravity experiences the extra dimensions, they could be of macroscopic size. Various scenarios have been suggested [9, 10].

Macroscopic extra dimensions imply deviations from Newton's law at such a scale. A year ago, millimeter scale deviations seemed plausible, and many theorists hoped that the higher-dimensional world was on the brink of discovery. Experiments were initiated [11]. Very recently, the results of the first definitive experiment have appeared [12], which indicate no deviation from Newton's law down to 218 $\mu$m. This poses a serious constraint for model-builders.[2]

## 2. Casimir Energies

Here we propose that a very tight constraint indeed emerges if we recognize that compact dimensions of size $a$ necessarily possess a quantum vacuum or Casimir energy of order $u(z)\sim a^{-4}$. That such energies are observable is confirmed by recent experiments [14]. These can be calculated in simple cases. Applequist and Chodos [15] found that the Casimir energy for the case of scalar field on a circle, $\mathcal{S}=S^1$, was

$$u_C=-\frac{3\zeta(5)}{64\pi^6a^4}=-\frac{5.056\times10^{-5}}{a^4}, \tag{6}$$

which needs only to be multiplied by 5 for graviton fluctuations. The general case of scalars on $\mathcal{S}=S^N$, $N$ odd, was considered by Candelas and Weinberg [16], who found that the Casimir energy was positive for $3\le N\le 19$, with a maximum at $N=13$ of $u_C=1.374\times10^{-3}/a^4$.

### 2.1. Green's Function Formalism

Let us remind the reader how these results are calculated, using the Green's function approach of Ref. [17]. We can write the Casimir energy density of a massless scalar field in a $M^4\times S^N$ manifold, the $N$-sphere having radius $a$ and volume $V_N$, as

$$\begin{aligned}u(a)&=V_N\langle T^{00}\rangle\\&=V_N\lim_{(x,y)\to(x',y')}\partial^0\partial'^0\mathrm{Im}\,G(x,y;x',y'),\end{aligned} \tag{7}$$

where the $x$ are the coordinates in the Minkowski space $M^4$, while the $S^N$ coordinates are denoted by $y$. For definiteness, we understand the point-splitting limit in Eq. (7) to be taken with a spacelike separation. Because of translational invariance in $x$, we can express $G$ as a four-dimensional Fourier transform,

$$G(x,y;x',y')=\int\frac{d^4k}{(2\pi)^4}e^{-ik_\mu(x-x')^\mu}g(y,y';k^\mu k_\mu), \tag{8}$$

in terms of which the vacuum energy can be simply expressed as

$$u(a)=-\frac{iV_N}{2(2\pi)^4}\int d^3k\int_c d\omega\,\omega^2g(y,y;k^2-\omega^2), \tag{9}$$

where the contour $c$ of the $\omega$ integration consists of $c_-$ and $c_+$, $c_+$ encircling the poles on the positive real axis in a clockwise sense, and $c_-$ encircling those on the negative real axis in a counterclockwise sense. See Fig. 1. The reduced Green's function satisfies

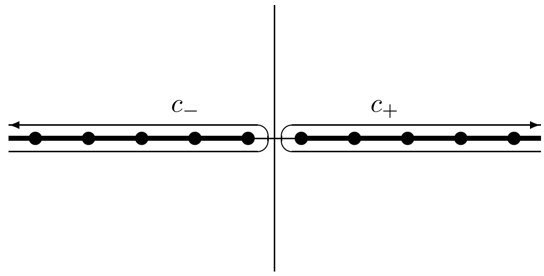


Figure 1: The $\omega$ plane for odd $N$ showing the Green's function contours in the complex $\omega$ plane. Shown schematically are the poles in the Green's function, and the branch cuts starting at $\beta=0$, where $\beta$ is given by Eq. (21). The corresponding branch points occur at $\omega=\pm\sqrt{k^2-(N-1)^2/4a^2}$. Note that if $k<(N-1)/2a$ the branch point and pole there lie on the imaginary axis. In that case, $c_+$ encloses the branch point on the positive imaginary axis, while $c_-$ encloses the branch point on the negative imaginary axis.

$$(\nabla_N^2+k^2-\omega^2)g(y,y';k^2-\omega^2)=-\delta(y-y'), \tag{10}$$

where $\nabla_N^2$ is the Laplacian on $S^N$ and $\delta(y-y')$ is the appropriate $\delta$ function.

In general we find $g$ for arbitrary $N$ by expanding in $N$-dimensional spherical harmonics:

$$\nabla_N^2Y_l^m(y)=-\frac{M_l^2}{a^2}Y_l^m(y), \tag{11}$$

where the eigenvalues and degeneracies are

$$M_l^2=l(l+N-1), \tag{12a}$$

$$D_l=\frac{(2l+N-1)(l+N-2)!}{(N-1)!\,l!}. \tag{12b}$$

Use of the generalized addition theorem for the hyperspherical harmonics,

$$\sum_m Y_l^m(y)Y_l^{m*}(y)=\frac{D_l}{V_N}, \tag{13}$$

[2] We might also mention short distance constraints on Yukawa-type corrections to the gravitational potential coming from Casimir measurements themselves [13].

leads to the following expression for the energy (9):

$$u(a) = -\frac{i}{(2\pi)^4}\int d^3k \int_{c_+} d\omega\,\omega^2 \times \sum_{l=0}^{\infty}\frac{D_l}{(M_l^2/a^2)+k^2-\omega^2}, \qquad (14)$$

where the integrand's dependence on $\omega^2$ has been used to combine the two parts of the $c$ contour to one, the right-hand contour $c_+$.

As is obvious from Eq. (14), the vacuum energy of the massless scalar in $M^4\times S^N$ is a linear sum of vacuum energies of massive scalars in 4 dimensions. The mode sum on $l$ diverges for $N>1$ and the momentum integrals diverge for all $N$. To obtain finite Casimir energies we can subtract off divergences identifiable as contact or cosmological terms from the outset. Because the $l$ sum is finite for the $N=1$ case, we consider that situation first.

#### 2.1.1. $N = 1$

In that case, the masses are $M_l^2 = l^2$, and the degeneracies are $D_0 = 1$, $D_{l\geq 1} = 2$, so we find for the mode sum, using the well-known cotangent identity,

$$\sum_{l=0}^{\infty}\frac{D_l}{M_l^2/a^2+k^2-\omega^2} = \frac{a\pi}{(k^2-\omega^2)^{1/2}}\coth\left[a\pi(k^2-\omega^2)^{1/2}\right] = \frac{a\pi}{(k^2-\omega^2)^{1/2}}\left(1+\frac{2}{e^{2\pi a(k^2-\omega^2)^{1/2}}-1}\right). \qquad (15)$$

This sum has been written as an asymptotic part plus a remainder. The asymptotic part produces an infinite "cosmological term" in the energy, and is either subtracted off completely, or regulated by inserting a cutoff $\omega_{\max}\sim b^{-1}$, $b$ presumably at the Planck scale, resulting in a cosmological energy density,[3]

$$u_{\rm cosmo}(a) = \frac{1}{(2\pi)^4}\int d^3k\, 2\int_k^\infty d\omega\,\omega^2\frac{a\pi}{(\omega^2-k^2)^{1/2}} = \frac{V_1}{80\pi^2 b^5}, \qquad (16)$$

where $V_1 = 2\pi a$ is the "volume" of a circle. Here we have taken an abrupt cutoff in $\omega$; however, any other technique also yields $u_{\rm cosmo}\sim V_1/b^5$. It is important to notice that the sum in Eq. (15) has only simple poles; however, the part that we identify as a cosmological term and subtract off has branch points at $\omega = \pm k$. For odd $N$, including the $N=1$ case, the branch cuts are drawn away from each other on the real $\omega$ axis out to $\pm\infty$ (see Fig. 1). The remainder of Eq. (15) produces the unique Casimir energy and is easily evaluated by (i) integrating over the $4\pi$ solid angle in the momentum element $d^3k$, (ii) distorting the contour $c_+$ to one lying along the imaginary $\omega$ axis, $\omega = i\zeta$ $(-\infty<\zeta<\infty)$, and (iii) replacing $\zeta$ and $k$ by plane polar coordinates, $k=\kappa\sin\theta$, $\zeta=\kappa\cos\theta$, and integrating first over $\theta$ and then over $\kappa$. The result is

$$u_{\rm Casimir} = -\frac{1}{64\pi^2a^4}\int_0^\infty(\kappa a)^4 d(\kappa a)^2\frac{2\pi}{\kappa a}\frac{1}{e^{2\pi\kappa a}-1} = -\frac{3\zeta(5)}{64\pi^6a^4} = -\frac{5.0558077\times10^{-5}}{a^4}. \qquad (17)$$

This is exactly the result first obtained by Appelquist and Chodos [15], and given in Eq. (6).

[3] If $b$ is the Planck scale, $b^{-4}\sim 10^{76}\,\mathrm{GeV}^4\sim 10^{108}\,\mathrm{GeV/cm}^3$, so even if $a/b\sim 1$ this is over a hundred orders of magnitude larger than the observed mass density of the universe, or of the current inferred value of the cosmological constant. This is the cosmological constant problem referred to in the Introduction.

| $N$ | $a^4u_N$ |
|---|---|
| 1 | $-5.0558077\times10^{-5}$ |
| 3 | $7.5687046\times10^{-5}$ |
| 5 | $4.2830382\times10^{-4}$ |
| 7 | $8.1588536\times10^{-4}$ |
| 9 | $1.1338947\times10^{-3}$ |
| 11 | $1.3293159\times10^{-3}$ |
| 13 | $1.3740262\times10^{-3}$ |
| 15 | $1.2524870\times10^{-3}$ |
| 17 | $9.5591579\times10^{-4}$ |
| 19 | $4.7935196\times10^{-4}$ |
| 21 | $-1.7990889\times10^{-4}$ |
| 23 | $-1.0231947\times10^{-3}$ |
| 25 | $-2.0509729\times10^{-3}$ |
| 27 | $-3.2631628\times10^{-3}$ |
| 29 | $-4.6593317\times10^{-3}$ |
| 31 | $-6.2388216\times10^{-3}$ |
| 33 | $-8.0008299\times10^{-3}$ |
| 35 | $-9.9444650\times10^{-3}$ |
| 37 | $-1.2068783\times10^{-2}$ |
| 39 | $-1.4372813\times10^{-2}$ |

Table 1: Casimir energy density $u_N(a)$ for a massless scalar in $M^4\times S^N$, with $N$ odd. The radius of the sphere is $a$.

#### 2.1.2. Odd $N$

The general odd-$N$ case can be calculated similarly, with the finite Casimir energies expressed in terms of sums of Riemann zeta functions. Details may be found in Ref. [17]. Numerical results are shown in Table 1.

### 2.2. Even $N$

The even dimensional case was much more subtle, because it was divergent. Kantowski and Milton [17] showed that the coefficient of the logarithmic divergence was unique, and adopting the Planck length as the natural cutoff, found

$$S^N,\ N\ \text{even}:\quad u_C^N = \frac{\alpha_N}{a^4}\ln\frac{a}{L_{\rm Pl}}, \qquad (18)$$

| $N$ | $\alpha_N$ |
|---|---|
| 2 | $-8.0413637 \times 10^{-5}$ |
| 4 | $-4.9923466 \times 10^{-4}$ |
| 6 | $-1.3144888 \times 10^{-3}$ |
| 8 | $-2.5052903 \times 10^{-3}$ |
| 10 | $-4.0355535 \times 10^{-3}$ |
| 12 | $-5.8734202 \times 10^{-3}$ |
| 14 | $-7.9931201 \times 10^{-3}$ |
| 16 | $-1.0373967 \times 10^{-2}$ |
| 18 | $-1.2999180 \times 10^{-2}$ |
| 20 | $-1.5844933 \times 10^{-2}$ |

Table 2: Coefficient $\alpha_N$ for the divergent logarithm for the Casimir energy for a massless scalar in $M^4 \times S^N$.

but $\alpha_N$ was always negative for scalars. Numerical results are found in Table 2.

#### 2.2.1. A Simple $\zeta$-Function Technique

The results above, found originally by the rigorous and physically transparent Green's function technique, can be quickly and easily reproduced by a simple $\zeta$-function method, which, as usual with such methods, sweeps divergence difficulties under the rug and does not reveal their interpretation as contact terms or cosmological-type terms. The scheme described in Ref. [18], however, is extremely simple to implement insofar as the Casimir energy is concerned. It is far simpler, in fact, than the method given in Refs. [16, 19, 20]. The starting point is the expression (14) for the energy

$$u = -\frac{ia^2}{(2\pi)^4}\int d^3k \int_{c_+} d\omega\, \omega^2 \sum_{m=1/2}^{\infty} \frac{D'_m}{m^2-\beta^2}. \tag{19}$$

Here

$$D'_m = \frac{2}{(N-1)!}\left[m^2 - \left(\frac{N-3}{2}\right)^2\right] \times \left[m^2 - \left(\frac{N-5}{2}\right)^2\right]\cdots\left[m^2 - \left(\frac{1}{2}\right)\right] m, \tag{20}$$

and

$$\beta^2 = [(N-1)/2]^2 + a^2\omega^2 - a^2k^2. \tag{21}$$

We regulate the integrals here by replacing the denominator by $(m^2-\beta^2)^{1+s}$, where ultimately $s$ will be taken to approach 0. If $s$ is large enough, we can exchange summation and integration, distort the $c_+$ contour to the imaginary $\omega$ axis, and introduce polar coordinates, $\omega = i\kappa\cos\theta$, $k = \kappa\sin\theta$. By first integrating over $\theta$, then over $\kappa$, we find

$$\begin{aligned} u(a) &= -\frac{a^2}{64\pi^2}\sum_{m=1/2}^{\infty} D'_m \\ &\times \int_0^\infty \frac{d\kappa^2\,\kappa^4}{[m^2+\kappa^2a^2-(N-1)^2/4]^{1+s}} \\ &= -\frac{1}{64\pi^2a^4}\sum_{m=1/2}^{\infty} D'_m\left(\frac{1}{s}-\frac{2}{s-1}+\frac{1}{s-2}\right) \\ &\times [m^2-(N-1)^2/4]^{2-s}. \end{aligned} \tag{22}$$

We next expand this in powers of $m$, and evaluate the $m$ sums according to

$$\sum_{m=1/2}^{\infty} m^z = (2^{-z}-1)\zeta(-z). \tag{23}$$

As $s \to 0$ the divergent terms are of the form $\alpha_N/(2a^4 s)$, where we identify $1/s$ with $\ln(a^2/L_{\rm Pl}^2)$ in Eq. (18). To isolate $\alpha_N$ we can multiply Eq. (22) by $2s$ and set $s=0$, not forgetting terms involving

$$s\zeta(1+2s) \to \frac{1}{2}. \tag{24}$$

This leads to the following easily implemented algorithm for $\alpha_N$:

1. Expand $D'_m(m^4 - 2m^2x + x^2)$ in powers of $m$, where $x = (N-1)^2/4$.
2. Make the replacement (23), that is replace $m^n$ by $(1/2^n - 1)\zeta(-n)$.
3. In the expansion of $D'_m$ replace $m^n$ by ($n$ is necessarily odd)
$$m^n \to x^{(n+5)/2}\frac{[(n-1)/2]!}{[(n+5)/2]!}. \tag{25}$$
4. Add the replacements in 2 and 3. [Steps 1 and 2 overlook terms of the form (24), step 3 includes them.]

### 2.3. Other Fields

In Ref. [18] Kantowski and Milton extended the analysis to vectors, tensors, fermions, and to massive particles, among which cases positive values of the (divergent) Casimir energy could be found. Some representative results for massless spin-1/2 fermions are shown in Table 3. In an unsuccessful attempt to find stable configurations, the analysis was extended to cases where the internal space was the product of spheres [21].

### 2.4. Proof of Covariance

We next note that the "Casimir energy" calculated here has the correct structure to be a cosmological constant. In Eq. (9) we gave an expression for the energy proportional to

$$u \propto \int d^3k \int_c d\omega\, \omega^2 g(y,y;k^2-\omega^2). \tag{26}$$

| Geometry $\mathcal{S}$ | Fermionic Casimir Energy |
|---|---|
| $S^1$ (u) | $\gamma = 2.02 \times 10^{-4}$ |
| $S^1$ (t) | $\gamma = -1.90 \times 10^{-4}$ |
| $S^2$ | $\alpha = -7.94 \times 10^{-4}$ |
| $S^3$ | $\gamma = 1.95 \times 10^{-4}$ |
| $S^4$ | $\alpha = -6.64 \times 10^{-3}$ |
| $S^5$ | $\gamma = -1.14 \times 10^{-4}$ |
| $S^6$ | $\alpha = -3.02 \times 10^{-2}$ |
| $S^7$ | $\gamma = 5.96 \times 10^{-5}$ |

Table 3: The Casimir energy for massless fermions in an $M^4 \times \mathcal{S}$ geometry.. We write $u = [\alpha \ln(a/L_{\rm Pl}) + \gamma]a^{-4}$, and give $\alpha$ for even internal dimension and $\gamma$ for odd, where $\alpha = 0$. For $S^1$ u denotes untwisted (periodic) while t twisted (antiperiodic) boundary conditions. The numbers are taken from Refs. [16, 18].

| Geometry $\mathcal{S}$ | Graviton Casimir Energy |
|---|---|
| $S^1$ (u) | $\gamma = -2.53 \times 10^{-4}$ |
| $S^1$ (t) | $\gamma = 2.37 \times 10^{-4}$ |
| $S^2$ | $\alpha = 1.70 \times 10^{-2}$ |
| $S^3$ | — |
| $S^4$ | $\alpha = -0.489$ |
| $S^5$ | — |
| $S^6$ | $\alpha = 5.10$ |
| $S^7$ | — |

Table 4: The Casimir energy due to graviton fluctuations for an $M^4 \times \mathcal{S}$ geometry. We write $u = [\alpha \ln(a/L_{\rm Pl}) + \gamma]a^{-4}$, and give $\alpha$ for even internal dimension and $\gamma$ for odd, where $\alpha = 0$. The entries marked with dashes have not been calculated. The numbers are taken from Ref. [27].

The $\omega^2$ came from the two time derivatives in $T^{00}$. If we were to calculate $T^{11}$ we would obtain

$$\langle T^{11}\rangle \propto \int d^3k \int_c d\omega\, \frac{1}{3}k^2 g(y,y;k^2-\omega^2), \tag{27}$$

since all three spatial directions are on the same footing. Recall that we may evaluate the integrals here by first making a Euclidean rotation, $\omega \to i\zeta$, and then adopt polar coordinates,

$$\zeta = \kappa\cos\theta, \quad k = \kappa\sin\theta, \tag{28}$$

so

$$T^{00} \propto -\int_0^{2\pi} d\theta \sin^2\theta\cos^2\theta = -\frac{\pi}{4}, \tag{29a}$$

$$T^{11} \propto \frac{1}{3}\int_0^{2\pi} d\theta\sin^4\theta = \frac{\pi}{4}. \tag{29b}$$

Thus the vacuum expectation value of the energy-momentum tensor has the required form (5) (of course, nothing else is possible, from relativistic covariance). [That this argument is not merely formal, but holds for the finite regulated terms as well, follows from the approach given in Sec. 2.2.1., for example, for even $N$.]

It is important to recognize that these Casimir energies correspond to a cosmological constant in our $3+1$ dimensional world, not in the extra compactified dimensions or "bulk." They constitute an effective source term in the 4-dimensional Einstein equations. That there is a correlation between the currently favored value of the cosmological constant and submillimeter-sized extra dimensions has been noted qualitatively before [22, 23].

### 2.5. Graviton Fluctuations

The goal, of course, in all these investigations was to include graviton fluctuations. However, it immediately became apparent that the results were gauge- and reparameterization-dependent unless the DeWitt-Vilkovisky formalism was adopted [24]. This was an extraordinarily difficult task. Some of the early papers on the unique effective action in simple cases are cited in Ref. [25]. Only in the past year has the general analysis for gravity appeared,[4] with results for a few special geometries [27]. Cho and Kantowski obtain the unique divergent part of the effective action for $\mathcal{S} = S^2$, $S^4$, and $S^6$, as polynomials in $\Lambda a^2$. (Unfortunately, once again, they are unable to find any stable configurations.) The results are shown in Table 4, for $\Lambda a^2 \sim G/a^2 \ll 1$. It will be noted that graviton fluctuations dominate matter fluctuations, except in the case of a large number of matter fields in a small number of dimensions. Of course, it would be very interesting to know the graviton fluctuation results for odd-dimensional spaces, but that seems to be a more difficult calculation; it is far easier to compute the divergent part than the finite part, which is all there is in odd-dimensional spaces.

These generic results may be applied to recent popular scenarios. For example, in the ADD scheme [9] only gravity propagates in the bulk, while the RS approach [10] has other bulk fields in a single extra dimension.

## 3. Constraint Arising from the Cosmological Constant

Let us now perform some simple estimates of the cosmological constant in these models. The data suggest a positive cosmological constant, so we can exclude those cases where the Casimir energy is negative. For the odd $N$ cases, where the Casimir energy is finite, let us write

$$S^N, \quad N\,\text{odd}: \quad u_C^N = \frac{\gamma_N}{a^4}, \tag{30}$$

so merely requiring that this be less than the critical density $\rho_c$ implies ($\gamma > 0$)

$$a \geq \gamma^{1/4} h_0^{-1/2} 67\,\mu\text{m} \approx \gamma^{1/4} 80\,\mu\text{m}, \tag{31}$$

[4] A few special cases were known earlier. Besides that of $\mathcal{S} = S^1$, the general six-dimensional background was considered by Cho and Kantowski [26], which includes $\mathcal{S} = S^1 \times S^1$ and $S^2$.

taking [28] $h_0 = 0.7$ (with about a 10–20% uncertainty). As seen in Table 5 these lower limits (for a single species) are still an order of magnitude below the experimental upper limit of about 200 $\mu$m [12]. Much tighter constraints appear if we use the divergent results for even dimensions. We have the inequality ($\alpha > 0$)

$$a \geq [\alpha \ln(a/L_{\rm Pl})]^{1/4} 80\,\mu\text{m}, \tag{32}$$

where we can approximate $(\ln a/L_{\rm Pl})^{1/4} \approx 2.9$. Again results are shown in Table 5, which rules out all but one of the gravity cases ($S^2$) given by Cho and Kantowski [27]. For matter fluctuations only [18], excluded are $N > 14$ for a single vector field and $N > 6$ for a single tensor field. (Fermions always have a negative Casimir energy in even dimensions.) Of course, it is possible to achieve cancellations by including various matter fields and gravity. In general the Casimir energy is obtained by summing over the species of field, which propagate in the extra dimensions,

$$u_{\rm tot} = \frac{1}{a^4}\sum_i [\alpha_i \ln(a/L_{\rm Pl}) + \gamma_i] \approx \frac{\gamma_{\rm eff}}{a^4}, \tag{33}$$

which leads to a lower limit according to Eq. (31). Presumably, if exact supersymmetry held in the extra dimensions (including supersymmetric boundary conditions), the Casimir energy would vanish, but this would seem to be difficult to achieve with *large* extra dimensions (1 mm corresponds to $2 \times 10^{-4}$ eV.)

| $\mathcal{S}$ | Gravity | Scalar | Fermion | Vector |
|---|---|---|---|---|
| $S^1$ (u) | * | * | 9.5 $\mu$m | — |
| $S^1$ (t) | 9.9 $\mu$m | 6.6 $\mu$m | * | — |
| $S^2$ | 84 $\mu$m | * | * | * |
| $S^3$ | — | 7.5 $\mu$m | 9.5 $\mu$m | — |
| $S^4$ | * | * | * | 77 $\mu$m |
| $S^5$ | — | 11.5 $\mu$m | * | — |
| $S^6$ | 350 $\mu$m | * | * | 110 $\mu$m |
| $S^7$ | — | 13.5 $\mu$m | 7.0 $\mu$m | — |

Table 5: The lower limit to the radius of the compact dimensions deduced from the requirement that the Casimir energy not exceed the critical density. The numbers shown are for a single species of the field type indicated. The dashes indicate cases where the Casimir energy has not been calculated, while asterisks indicate (phenomenologically excluded) cases where the Casimir energy is negative. The vector results for even-$N$ are taken from Ref. [18]. (We should note that a general recipe for calculating the odd-$N$ terms for vectors and tensors is given in Ref. [19], but results are not explicit, and require knowledge of the "polylogarithmic-exponential" function. Moreover, the Casimir energies they find are complex. The method given in Ref. [18] has, in contrast, no problems with tachyons, and would give real energies. Explicit numbers were given in Ref. [29] for the various components of gravity without the necessary Vilkovisky-DeWitt correction. However, the transverse vector part cannot be extracted without further calculation.)

## 4. Conclusions

It seems to be commonly believed that submillimeter tests of gravity put no limits on the size of extra dimensions if $N > 2$. This is because of the relation of the size $R$ of the extra dimensions in the ADD scheme to the fundamental $4 + N$ gravity scale $M$ [30]:

$$R \sim \frac{1}{M}\left(\frac{M_{\rm Pl}}{M}\right)^{2/N}, \tag{34}$$

where $M_{\rm Pl} = 1/L_{\rm Pl} = 1.2 \times 10^{16}$ TeV is the usual Planck mass. Moreover, the supernova limits on ADD extra dimensions (due to production of Kaluza-Klein gravitons) become rapidly smaller with increase in $N$ [31]:

$$N = 2: \quad R < 0.9 \times 10^{-4}\,\text{mm}, \tag{35a}$$

$$N = 3: \quad R < 1.9 \times 10^{-7}\,\text{mm}. \tag{35b}$$

Thus direct tests of Newton's law are not competitive. However, the resulting Casimir contribution to the cosmological constant would be enormous for such small compactified regions, and it would seem impossible to naturally resolve this problem.

The situation at first glance seems rather different with the RS scenario. In the original scheme, gravity is localized in the "Planck brane," while the standard-model particles are confined to the "TeV brane." As a consequence, it might appear that the quantum fluctuations of both brane and bulk fields are negligible [32]. It has been stated that the cosmological constant becomes exponentially small as the brane separation becomes large [33]. However, this is at the "classical level," without bulk fluctuations; explicit considerations show that quantum effects give rise to a large cosmological constant, of order of that given by Eq. (3), unless an appeal is made to fine tuning [34]. Moreover, if the scenario is extended so that the world brane contains compactified dimensions in which gravity lives [35], the constraints we deduce here directly apply.

There have been a number of proposals in which either in string [36, 37] or brane [38] contexts the cosmological constant can be made to vanish. These solutions may not be altogether natural, and may indeed require fine tuning [39]. (Moreover, the suggestions either do not include gravitational fluctuations, or ignore the problem of gauge and parameterization dependence.) Such ideas could explain the vanishing of $u_{\rm cosmo}$, or of a corresponding energy arising at the electroweak symmetry breaking scale, $\Lambda_{\rm EW} \sim 1$ TeV,

$$u_{\rm EW} \sim \Lambda_{\rm EW}^4 \sim 10^{53}\text{GeV/cm}^3, \tag{36}$$

due to fluctuations in standard model fields, but they would presumably not lead to the simultaneous vanishing of the Casimir energy. Indeed Sundrum in Ref. [36] obtains a small cosmological constant based on a narrow window in allowed graviton compositeness, or string, scale, $m_{\rm st}$,

$$10\,\mu\text{m} < \frac{1}{m_{\rm st}} < 1\,\text{cm}. \qquad (37)$$

Theoretical and experimental limits this past year have nearly closed this window. Clearly, the ideas expressed here provide a stringent constraint for model builders.

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## References

[1] S. Weinberg, *Rev. Mod. Phys.* **61**, 1 (1989); *Phys. Rev. D* **61**, 103505 (2000); talk at Dark Matter 2000, Marina del Rey, CA, 2000, astro-ph/0005265.

[2] P.M. Garnavich et al., *Astrophys. J. Lett.* **493**, L53 (1998); A. G. Reiss et al., *Astron. J.* **116**, 1009 (1998).

[3] S. Perlmutter et al., *Astrophys. J.* **517**, 565 (1999).

[4] The BOOMERANG collaboration: P. de Bernardis et al., *Nature* **404**, 955 (2000); A. E. Lange et al., *Phys. Rev. D* **63**, 042001 (2001); C. B. Netterfield et al., astro-ph/0104460.

[5] The MAXIMA collaboration: A. Balbi et al., *Astrophys. J. Lett.* **545**, L1 (2000); S. Hanany et al., *Astrophys. J. Lett.* **545**, L5 (2000).

[6] The DASI collaboration: C. Pryke et al., astro-ph/0104490.

[7] K. A. Milton and Y. J. Ng, *Phys. Rev. E* **57**, 5504 (1998); I. Brevik, V. N. Marachevsky, and K. A. Milton, *Phys. Rev. Lett.* **82**, 3948 (1999); G. Barton, *J. Phys. A* **32**, 525 (1999); M. Bordag, K. Kirsten, and D. Vassilevich, *Phys. Rev. D* **59**, 085011 (1999); J. S. Høye and I. Brevik, *J. Stat. Phys.* **100**, 223 (2000).

[8] T. Kaluza, *Sitz. Preuss. Akad. Wiss. Phys. Math.* **K1**, 966 (1921); O. Klein, *Nature* **118**, 516 (1926); *Z. Phys.* **37**, 895 (1926).

[9] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phys. Lett. B* **429**, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phys. Lett. B* **436**, 257 (1998).

[10] L. Randall and R. Sundrum, *Phys. Rev. Lett.* **83**, 3370 (1999); **83**, 4690 (1999).

[11] J. C. Long, H. W. Chan, and J. C. Price, *Nucl. Phys. B* **539**, 23 (1999); J. C. Long, A. B. Churnside, and J. C. Price, in *Proceedings of the 9th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Gravitation, and Relativistic Field Theories (MG9), Rome, 2–9 July 2000*, hep-ph/0009062.

[12] C. D. Hoyle, U. Schmidt, B. R. Heckel, E. G. Adelberger, J. H. Gundlach, D. J. Kapner, and H. E. Swanson, *Phys. Rev. Lett.* **86**, 1418 (2001).

[13] M. Bordag, B. Geyer, G. L. Klimchitskaya, and V. M. Mostepanenko, *Phys. Rev. D* **58**, 075003 (1998); **60**, 055004 (1999); **62**, 011701(R) (2000); V. M. Mostepaneko and M. Novello, *ibid.* **63**, 115003 (2001); E. Fischbach, D. E. Krause, V. M. Mostepanenko, M. Novello, hep-ph/0106331.

[14] S. K. Lamoreaux, *Phys Rev. Lett.* **78**, 5 (1997); **81**, 5475(E) (1998); U. Mohideen and A. Roy, *ibid.* **81**, 4549 (1998); **83**, 3341 (1999); A. Roy, C.-Y. Lin, and U. Mohideen, *Phys. Rev. D* **60**, R111101 (1999); B. W. Harris, F. Chen, and U. Mohideen, *Phys. Rev. A* **62**, 052109 (2000); H. B. Chan, V. A. Aksyuk, R. N. Kleiman, D. J. Bishop, and F. Capasso, *Science* **291**, 1941 (2001) (10.1126/science.1057984).

[15] T. Appelquist and A. Chodos, *Phys. Rev. Lett.* **50**, 141 (1983); *Phys. Rev. D* **28**, 772 (1983).

[16] P Candelas and S. Weinberg, *Nucl. Phys. B* **237**, 397 (1984).

[17] R. Kantowski and K. A. Milton, *Phys. Rev. D* **35**, 549 (1987).

[18] R. Kantowski and K. A. Milton, *Phys. Rev. D* **36**, 3712 (1987).

[19] A. Chodos and E. Myers, *Ann. Phys. (N.Y.)* **156**, 412 (1984); *Phys. Rev. D* **31**, 3064 (1985).

[20] E. Myers, *Phys. Rev. D* **33**, 1563 (1986).

[21] D. Birmingham, R. Kantowski, and K. A. Milton, *Phys. Rev. D* **38**, 1809 (1988).

[22] D. B. Kaplan and M. B. Wise, *JHEP* **08**, 037 (2000); T. Banks, *Nucl. Phys. B* **309**, 493 (1988); S. R. Beane, *Phys. Lett. B* **358**, 203 (1995); *Gen. Rel. Grav.* **29**, 945 (1997).

[23] J.-W. Chen, M. A. Luty, and E. Pontón, *JHEP* **09**, 012 (2000).

[24] G. A. Vilkovisky, in *Quantum Theory of Gravity*, ed. S. C. Christensen, Hilger, Bristol, 1984; *Nucl. Phys. B* **234**, 125 (1984); A. O. Barvinsky and G. A. Vilkovisky, *Phys. Rep.* **119**, 1 (1985); B. S. DeWitt, in *Quantum Field Theory and Quantum Statistics*, ed. I. A. Batalin, C. J. Isham, and G. A. Vilkovisky, Hilger, Bristol, 1987.

[25] S. R. Huggins, G. Kunstatter, H. P. Leivo, and D. J. Toms, *Phys. Rev. Lett.* **58**, 296 (1987); *Nucl. Phys. B* **301**, 627 (1988); I. L. Buchbinder and S. D. Odintsov, *Yad. Fiz.* **46**, 1233 (1987); P. M. Lavrov, S. D. Odintsov and I. V. Tyutin, *Mod. Phys. Lett. A* **3**, 1273 (1988); I. L. Buchbinder, P. M. Lavrov and S. D. Odintsov, *Nucl. Phys. B* **308**, 191 (1988); S. D. Odintsov, *Phys. Lett. B* **215**, 483 (1988); **262**, 394 (1991).

[26] H. T. Cho and R. Kantowski, *Phys. Rev. Lett.* **67**, 422 (1991); **75**, 2630(E) (1995).

[27] H. T. Cho and R. Kantowski, *Phys. Rev. D* **62**, 124003 (2000).

[28] Particle Data Group, *Eur. Phys. J. C* **15**, 1 (2000); W. J. Freedman et al., astro-ph/0012376, *Astrophys. J.*, to be published; D. Branch, *Annu. Rev. Astron. Astrophys.* **36**, 17 (1998), astro-ph/9801065; and references therein.

[29] M. H. Sarmadi, *Nucl. Phys. B* **263**, 187 (1986).

[30] N. Arkani-Hamed, S. Dimopoulos, and G. Divali, *Phys. Rev. D* **59**, 086004 (1999); G. F. Guidice, R. Rattazzi, and J. D. Wells, *Nucl. Phys. B* **544** 3 (1999); E. A. Mirabelli, M. Perelstein, and M. E. Peshkin, *Phys. Rev. Lett.* **82**, 2236 (1999).

[31] S. Hannestad and G. Raffelt, hep-ph/0103201; V. Barger, T. Han, C. Kao and R. J. Zhang, *Phys. Lett. B* **461**, 34 (1999); S. Cullen and M. Perelstein, *Phys. Rev. Lett.* **83**, 268 (1999).

[32] W. D. Goldberger and I. Z. Rothstein, *Phys. Lett. B* **475**, 275 (2000); I. Brevik, K. A. Milton, S. Nojiri, and S. D. Odintsov, *Nucl. Phys. B* **599**, 305 (2001).

[33] S.-H. H. Tye and I. Wasserman, *Phys. Rev. Lett.* **86**, 1682 (2001).

[34] S. Nojiri, O. Obregon, and S. D. Odintsov, hep-th/0105300.

[35] Z. Chako, P. J. Fox, A. E. Nelson, and N. Weiner, hep-ph/0106343.

[36] R. Sundrum, *JHEP* **9907**, 001 (1999).

[37] S. Kachru and E. Silverstein, *JHEP* **01**, 004 (1999); S. Kachru, J. Kumar, and E. Silverstein, *Phys. Rev. D* **59**, 106004 (1999); S. Kachru, M. Schulz, and E. Silverstein, *Phys. Rev. D* **62**, 045021 (2000).

[38] N. Arkani-Hamed, S. Dimopoulos, N. Kaloper, and R. Sundrum, *Phys. Lett. B* **480**, 193 (2000); S. Förste, Z. Lalak, S. Lavignac, and H. P. Nilles, *Phys. Lett. B* **481**, 360 (2000).

[39] S. Förste, Z. Lalak, S. Lavignac, and H. P. Nilles, *JHEP* **09**, 034 (2000).

[40] K. A. Milton, R. Kantowski, C. Kao, and Yun Wang, "Constraints on Extra Dimensions from Cosmological and Terrestrial Measurements," OKHEP–01–03, hep-ph/0105250.

[41] K. A. Milton, *The Casimir Effect: Physical Manifestations of Zero-Point Energy*, World Scientific, Singapore, in press.