$\mathcal{PT}$-Symmetric Quantum Electrodynamics

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Abstract

The Hamiltonian for quantum electrodynamics becomes non-Hermitian if the unrenormalized electric charge e is taken to be imaginary. However, if one also specifies that the potential A^μ in such a theory transforms as a pseudovector rather than a vector, then the Hamiltonian becomes $\mathcal{PT}$ symmetric. The resulting non-Hermitian theory of electrodynamics is the analog of a spinless quantum field theory in which a pseudoscalar field φ has a cubic self-interaction of the form $i\varphi^3$. The Hamiltonian for this cubic scalar field theory has a positive spectrum, and it has recently been demonstrated that the time evolution of this theory is unitary. The proof of unitarity requires the construction of a new operator called $\mathcal{C}$, which is then used to define an inner product with respect to which the Hamiltonian is self-adjoint. In this paper the corresponding $\mathcal{C}$ operator for non-Hermitian quantum electrodynamics is constructed perturbatively. This construction demonstrates the unitarity of the theory. Non-Hermitian quantum electrodynamics is a particularly interesting quantum field theory model because it is asymptotically free.

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I. INTRODUCTION

It is common wisdom that the Hamiltonian that defines a quantum theory should be Hermitian $H = H^\dagger$, where the symbol $\dagger$, which indicates Dirac Hermitian conjugation, represents the combined operations of complex conjugation and matrix transposition. There are two reasons given for requiring that the Hamiltonian be Hermitian: First, the condition $H = H^\dagger$ guarantees that the energy eigenvalues of H will be real. Second, this condition guarantees that time evolution will be unitary; that is, that probability will be conserved.

However, in the past few years it has become clear that the requirements of spectral positivity and unitarity can be met even if the Hamiltonian is not Hermitian in the Dirac sense. The first non-Hermitian Hamiltonian for which these two properties were verified was the quantum-mechanical model

$$H = p^2 + x^2(ix)^\epsilon \quad (\epsilon \geq 0). \tag{1}$$

It was observed in 1998 that the spectrum of this class of Hamiltonians was positive and discrete [1] and it was conjectured that spectral positivity was a consequence of the invariance of H under the combination of the space-reflection operator $\mathcal{P}$ and the time-inversion operator $\mathcal{T}$. Three years later, a proof of spectral positivity was given [2]. Then, in 2002 it was shown that the Hamiltonian in (1) defines a unitary time evolution [3]. Specifically, it was demonstrated that if the $\mathcal{PT}$ symmetry of a non-Hermitian Hamiltonian is unbroken, then it is possible to construct a new operator called $\mathcal{C}$ that commutes with the Hamiltonian H. The Hilbert space inner product with respect to the $\mathcal{CPT}$ adjoint has a positive norm and the time evolution operator e^{iHt} is unitary. Thus, from this quantum-mechanical study it is clear that Dirac Hermiticity of the Hamiltonian is not a necessary requirement of a quantum theory; unbroken $\mathcal{PT}$ symmetry is sufficient to guarantee that the spectrum of H is real and positive and that the time evolution is unitary.

The construction of the $\mathcal{C}$ operator in Ref. [3] was the key step in showing that the non-Hermitian Hamiltonian (1) exhibits unitary time evolution. However, the difficulty with the construction given in Ref. [3] is that the calculation of the $\mathcal{C}$ operator required as input all the coordinate-space eigenvectors of the Hamiltonian. While this information is, in principle, available in quantum mechanics, it is hardly available for a quantum field theory because there is no simple analog of the coordinate-space Schrödinger equation. Thus, the analysis in Ref. [3] does not extend easily to quantum field theory.

However, it was recently shown that a perturbative construction of $\mathcal{C}$ that does not require the eigenfunctions of the Hamiltonian is possible for the case of a scalar quantum field theory with a cubic self-interaction of the form $i\phi^3$ [4]. This result is particularly important because this quantum field theory has already appeared in the literature in studies of the Lee-Yang edge singularity [5] and in Reggeon field theory [6]. The construction of the $\mathcal{C}$ operator for the $i\phi^3$ field theory shows that this quantum field theory is a fully acceptable unitary quantum theory and not just an interesting but unrealistic mathematical curiosity.

Furthermore, an exact construction of the $\mathcal{C}$ operator [7] was carried out for the Lee model, a cubic quantum field theory in which mass, wave-function, and coupling-constant renormalization can be done exactly [8]. The construction of the $\mathcal{C}$ operator for the Lee model explains a long-standing puzzle. It is known that there is a critical value of the renormalized coupling constant g for the Lee model and that when g exceeds this critical value, the unrenormalized coupling constant becomes pure imaginary, and hence the Hamiltonian becomes non-Hermitian. As a consequence, a ghost state having negative Hermitian norm appears when $g > g_{\rm crit}$, and the presence of this ghost state causes the S matrix to be nonunitary. By constructing the $\mathcal{C}$ operator we can reinterpret the Hilbert space for the theory. By using a $\mathcal{CPT}$ inner product, the ghost state now has a positive norm and the Lee model becomes a consistent unitary quantum field theory. This physical reinterpretation of the Lee model was anticipated by F. Kleefeld in a beautiful series of papers [9].

Recently, additional progress was made in understanding the $\mathcal{C}$ operator in the context of an $ig\phi^3$ quantum field theory. It was shown that $\mathcal{C}$ transforms as a scalar under the action of the homogeneous Lorentz group [10]. In this paper it was argued that because the Hamiltonian H_0 for the unperturbed theory ($g = 0$) commutes with the parity operator $\mathcal{P}$, the intrinsic parity operator $\mathcal{P}_{\rm I}$ in the noninteracting theory transforms as a Lorentz scalar. (The *intrinsic* parity operator $\mathcal{P}_{\rm I}$ and the parity operator $\mathcal{P}$ have the same effect on the fields, except that $\mathcal{P}_{\rm I}$ does not reverse the sign of the spatial argument of the field.) When the coupling constant g is nonzero, the parity symmetry of H is broken and $\mathcal{P}_{\rm I}$ is no longer a scalar. However, $\mathcal{C}$ *is* a scalar. Since $\lim_{g\to 0}\mathcal{C} = \mathcal{P}_{\rm I}$, one can interpret the $\mathcal{C}$ operator as the complex extension of the intrinsic parity operator when the imaginary coupling constant is turned on.

In this paper we examine $\mathcal{PT}$-symmetric quantum electrodynamics, a non-Hermitian quantum field theory that is much more interesting than an $i\phi^3$ field theory. Unlike the scalar $i\phi^3$ field theory, $\mathcal{PT}$-symmetric quantum electrodynamics possesses many of the features of conventional quantum electrodynamics, including Abelian gauge invariance. Two earlier preliminary studies of this theory have already been published [11, 12]. The advance reported in the present paper is the construction of the $\mathcal{C}$ operator to leading order in perturbation theory for this remarkable theory. Our construction provides strong evidence that $\mathcal{PT}$-symmetric quantum electrodynamics is a viable and consistent unitary quantum field theory.

While $\mathcal{PT}$-symmetric quantum electrodynamics is similar to an $i\phi^3$ field theory because its interaction Hamiltonian is cubic and its coupling constant is pure imaginary, this quantum field theory is especially interesting because, like a $\mathcal{PT}$-symmetric $-\phi^4$ scalar quantum field theory in four dimensions, $\mathcal{PT}$-symmetric electrodynamics is asymptotically free [13]. The only asymptotically free quantum field theories described by Hermitian Hamiltonians are those that possess a *non-Abelian* gauge invariance; $\mathcal{PT}$ symmetry allows for new kinds of asymptotically free theories that do not have to possess a non-Abelian gauge invariance.

II. $\mathcal{PT}$-SYMMETRIC QUANTUM ELECTRODYNAMICS

In order to formulate a Lorentz covariant quantum field theory one begins by specifying the Lorentz transformation properties of the quantum fields under the proper orthochronous Lorentz group. [For example, one can specify that the field $\phi(\mathbf{x},t)$ transforms as a scalar.] In addition, one is free to specify the transformation properties of the fields under parity reflection. [For example, one can specify that $\phi(\mathbf{x},t)$ transforms as a scalar, so that it does not change sign under $\mathcal{P}$, or that it transforms as a pseudo-scalar, so that it changes sign under $\mathcal{P}$.] Having fully specified the transformation properties of the fields, one then formulates the (scalar) Lagrangian in terms of these fields.

A non-Hermitian but $\mathcal{PT}$-symmetric version of electrodynamics can be constructed by assuming that the four-vector potential transforms as an *axial* vector [12]. As a consequence, the electromagnetic fields transform under parity reflection like

$$\mathcal{P}: \quad \mathbf{E}\to\mathbf{E}, \quad \mathbf{B}\to-\mathbf{B}, \quad \mathbf{A}\to\mathbf{A}, \quad A^0\to -A^0. \tag{2}$$

Under time reversal, the transformations are assumed to be conventional:

$$\mathcal{T}: \quad \mathbf{E} \to \mathbf{E}, \quad \mathbf{B} \to -\mathbf{B}, \quad \mathbf{A} \to -\mathbf{A}, \quad A^0 \to A^0. \tag{3}$$

The Lagrangian of the theory then possesses an imaginary coupling constant in order that it be invariant under the product of these two symmetries:

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}\psi^\dagger\gamma^0\gamma^\mu\tfrac{1}{i}\partial_\mu\psi + \tfrac{1}{2}m\psi^\dagger\gamma^0\psi + ie\psi^\dagger\gamma^0\gamma^\mu\psi A_\mu. \tag{4}$$

The corresponding Hamiltonian density is then

$$\mathcal{H} = \tfrac{1}{2}(E^2 + B^2) + \psi^\dagger\left[\gamma^0\gamma^k\left(-i\nabla_k + ieA_k\right) + m\gamma^0\right]\psi. \tag{5}$$

The electric current appearing in both the Lagrangian and Hamiltonian densities, $j^\mu = \psi^\dagger\gamma^0\gamma^\mu\psi$, transforms conventionally under both $\mathcal{P}$ and $\mathcal{T}$:

$$\mathcal{P}j^\mu(\mathbf{x},t)\mathcal{P} = \begin{pmatrix} j^0 \\ -\mathbf{j} \end{pmatrix}(-\mathbf{x},t), \tag{6a}$$

$$\mathcal{T}j^\mu(\mathbf{x},t)\mathcal{T} = \begin{pmatrix} j^0 \\ -\mathbf{j} \end{pmatrix}(\mathbf{x},-t). \tag{6b}$$

Just as in the case of ordinary quantum electrodynamics, $\mathcal{PT}$-symmetric electrodynamics has an Abelian gauge invariance. In this paper we choose to work in the Coulomb gauge, $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$, so the nonzero canonical equal-time commutation relations are

$$\{\psi_a(\mathbf{x},t), \psi_b^\dagger(\mathbf{y},t)\} = \delta_{ab}\delta(\mathbf{x}-\mathbf{y}), \tag{7a}$$

$$[A_i^T(\mathbf{x}), E_j^T(\mathbf{y})] = -i\left[\delta_{ij} - \frac{\nabla_i\nabla_j}{\nabla^2}\right]\delta(\mathbf{x}-\mathbf{y}), \tag{7b}$$

where T denotes the transverse part,

$$\boldsymbol{\nabla}\cdot\mathbf{A}^T = \boldsymbol{\nabla}\cdot\mathbf{E}^T = 0. \tag{8}$$

In the following, the symbols $\mathbf{E}$ and $\mathbf{B}$ represent the transverse parts of the electromagnetic fields, so

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \boldsymbol{\nabla}\cdot\mathbf{B} = 0. \tag{9}$$

III. CALCULATION OF THE $\mathcal{C}$ OPERATOR

As in quantum-mechanical systems and scalar quantum field theories, we seek a $\mathcal{C}$ operator in the form [4]

$$\mathcal{C} = \mathrm{e}^Q\mathcal{P}, \tag{10}$$

where $\mathcal{P}$ is the parity operator, and our objective will be to calculate the operator Q [14]. Because $\mathcal{C}$ must satisfy the three defining properties

$$\mathcal{C}^2 = 1, \tag{11a}$$

$$[\mathcal{C}, \mathcal{PT}] = 0, \tag{11b}$$

$$[\mathcal{C}, H] = 0, \tag{11c}$$

we infer from Eq. (11a) that

$$Q = -\mathcal{P}Q\mathcal{P}, \tag{12a}$$

and because $\mathcal{P}\mathcal{T} = \mathcal{T}\mathcal{P}$, we infer from (11b) that

$$Q = -\mathcal{T}Q\mathcal{T}. \tag{12b}$$

The two equations (11a) and (11b) can be thought of as kinematical constraints on Q.

The third equation (11c), which can be thought of as a dynamical condition on Q, allows us to determine Q perturbatively. If we separate the interaction part of the Hamiltonian from the free part,

$$H = H_0 + eH_1, \tag{13}$$

and seek Q in the form of a power series

$$Q = eQ_1 + e^2 Q_2 + \cdots, \tag{14}$$

then the first contribution to the Q operator is determined by

$$[Q_1, H_0] = 2H_1. \tag{15}$$

As in previous studies of cubic quantum theories, the second correction commutes with the Hamiltonian,

$$[Q_2, H_0] = 0, \tag{16}$$

and Eq. (14) reduces to a series in odd powers of e,

$$Q = eQ_1 + e^3 Q_3 + \cdots, \tag{17}$$

which illustrates the virtue of the exponential representation (10).

To use Eq. (15) to determine the operator Q_1, we construct the most general nonlocal *ansatz* for the operator Q_1 in terms of the sixteen independent Dirac tensors. There is no condition of gauge invariance on this operator because we have chosen to work in the Coulomb gauge. There are sixteen tensor functions in principle, which we take to be defined by

$$\begin{aligned} Q_1 = \int d\mathbf{x}\, d\mathbf{y}\, d\mathbf{z} \Big\{ & \left[f_+^{kl}(\mathbf{x},\mathbf{y},\mathbf{z})E^k(\mathbf{x}) + f_-^{kl}(\mathbf{x},\mathbf{y},\mathbf{z})B^k(\mathbf{x})\right] \psi^\dagger(\mathbf{y})\gamma^0\gamma^l\psi(\mathbf{z}) \\ & + \left[g_+^{k}(\mathbf{x},\mathbf{y},\mathbf{z})E^k(\mathbf{x}) + g_-^{k}(\mathbf{x},\mathbf{y},\mathbf{z})B^k(\mathbf{x})\right] \psi^\dagger(\mathbf{y})\gamma^0\gamma^5\psi(\mathbf{z}) \\ & + \left[h_+^{k}(\mathbf{x},\mathbf{y},\mathbf{z})E^k(\mathbf{x}) + h_-^{k}(\mathbf{x},\mathbf{y},\mathbf{z})B^k(\mathbf{x})\right] \psi^\dagger(\mathbf{y})\gamma^5\psi(\mathbf{z}) \\ & + \left[j_+^{kl}(\mathbf{x},\mathbf{y},\mathbf{z})E^k(\mathbf{x}) + j_-^{kl}(\mathbf{x},\mathbf{y},\mathbf{z})B^k(\mathbf{x})\right] \psi^\dagger(\mathbf{y})\gamma^l\psi(\mathbf{z}) \\ & + \left[F_+^{kl}(\mathbf{x},\mathbf{y},\mathbf{z})B^k(\mathbf{x}) + F_-^{kl}(\mathbf{x},\mathbf{y},\mathbf{z})E^k(\mathbf{x})\right] \psi^\dagger(\mathbf{y})\gamma^0\gamma^5\gamma^l\psi(\mathbf{z}) \\ & + \left[G_+^{k}(\mathbf{x},\mathbf{y},\mathbf{z})B^k(\mathbf{x}) + G_-^{k}(\mathbf{x},\mathbf{y},\mathbf{z})E^k(\mathbf{x})\right] \psi^\dagger(\mathbf{y})\gamma^0\psi(\mathbf{z}) \\ & + \left[H_+^{k}(\mathbf{x},\mathbf{y},\mathbf{z})B^k(\mathbf{x}) + H_-^{k}(\mathbf{x},\mathbf{y},\mathbf{z})E^k(\mathbf{x})\right] \psi^\dagger(\mathbf{y})\psi(\mathbf{z}) \\ & + [J_+^{kl}(\mathbf{x},\mathbf{y},\mathbf{z})B^k(\mathbf{x}) + J_-^{kl}(\mathbf{x},\mathbf{y},\mathbf{z})E^k(\mathbf{x})]\psi^\dagger(\mathbf{y})\gamma^5\gamma^l\psi(\mathbf{z}) \Big\}. \end{aligned} \tag{18}$$

In Eq. (18) we have taken into account the fact that the electric and magnetic fields are transverse, $\boldsymbol{\nabla}\cdot\mathbf{E}=\boldsymbol{\nabla}\cdot\mathbf{B}=0$ [see Eq. (9)]. The parity constraint (12a) is satisfied because $f_\pm$, $g_\pm$, $\cdots$, are respectively even and odd functions:

$$f_\pm(\mathbf{x},\mathbf{y},\mathbf{z})=\pm f_\pm(-\mathbf{x},-\mathbf{y},-\mathbf{z}). \tag{19}$$

We will see that the time-reversal constraint (12b) is automatically satisfied by Q_1 in (18).

The solution of Eq. (15) is obtained by using the canonical commutation relations (7a) and (7b), which imply that

$$\left[E^k(\mathbf{x}),\frac{1}{2}\int d\mathbf{w}B^2(\mathbf{w})\right] = i(\boldsymbol{\nabla}\times\mathbf{B})_k(\mathbf{x}), \tag{20a}$$

$$\left[B^k(\mathbf{x}),\frac{1}{2}\int d\mathbf{w}E^2(\mathbf{w})\right] = -i(\boldsymbol{\nabla}\times\mathbf{E})_k(\mathbf{x}), \tag{20b}$$

$$\begin{aligned}\Bigg[\int d\mathbf{y}\,d\mathbf{z}\,\phi(\mathbf{y},\mathbf{z})\psi^\dagger(\mathbf{y})\Gamma\psi(\mathbf{z}),\int d\mathbf{w}\,\psi^\dagger(\mathbf{w})\gamma^0\gamma^k\frac{1}{i}\nabla_k\psi(\mathbf{w})\Bigg]&\\ =\frac{i}{2}\int d\mathbf{y}\,d\mathbf{z}\big[(\nabla_k^z+\nabla_k^y)\phi(\mathbf{y},\mathbf{z})\psi^\dagger(\mathbf{y})\{\Gamma,\gamma^0\gamma^k\}\psi(\mathbf{z})&\\ +(\nabla_k^z-\nabla_k^y)\phi(\mathbf{y},\mathbf{z})\psi^\dagger(\mathbf{y})[\Gamma,\gamma^0\gamma^k]\psi(\mathbf{z})\big],&\end{aligned} \tag{20c}$$

$$\begin{aligned}\Bigg[\int d\mathbf{y}\,d\mathbf{z}\,\phi(\mathbf{y},\mathbf{z})\psi^\dagger(\mathbf{y})\Gamma\psi(\mathbf{z}),m\int d\mathbf{w}\,\psi^\dagger(\mathbf{w})\gamma^0\psi(\mathbf{w})\Bigg]&\\ =m\int d\mathbf{y}\,d\mathbf{z}\,\phi(\mathbf{y},\mathbf{z})\psi^\dagger(\mathbf{y})[\Gamma,\gamma^0]\psi(\mathbf{z}).&\end{aligned} \tag{20d}$$

There are sixteen resulting equations for the tensor coefficients, which break up into two independent sets of eight equations each. Since there is only one inhomogeneous term, this means that the coefficients that satisfy the set of equations with no driving term must vanish. The remaining equations are most conveniently written in momentum space, where the Fourier transform is defined by

$$\tilde{f}(\mathbf{p})=\int d\mathbf{x}\,\mathrm{e}^{-i\mathbf{p}\cdot\mathbf{x}}f(\mathbf{x}). \tag{21}$$

If the momenta corresponding to the coordinates $\mathbf{x},\mathbf{y},\mathbf{z}$ are $\mathbf{p},\mathbf{q},\mathbf{r}$, then as a result of translational invariance there is an overall momentum-conserving delta function, which sets $\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0}$. Using dyadic notation, it is not hard to show that these equations are, in terms of the two independent vectors $\mathbf{p}$ and $\mathbf{t}=\mathbf{r}-\mathbf{q}$, given by

$$\mathbf{p}\times\tilde{\mathbf{g}}_-+\tilde{\mathbf{J}}_-\cdot\mathbf{t}-2m\tilde{\mathbf{h}}_+ = \mathbf{0}, \tag{22a}$$

$$\mathbf{p}\times\tilde{\mathbf{h}}_++\tilde{\mathbf{F}}_+\cdot\mathbf{p}+2m\tilde{\mathbf{g}}_- = \mathbf{0}, \tag{22b}$$

$$\mathbf{p}\times\tilde{\mathbf{j}}_--i\tilde{\mathbf{J}}_-\times\mathbf{p}-\tilde{\mathbf{G}}_-\mathbf{t}-2m\tilde{\mathbf{f}}_+ = \mathbf{0}, \tag{22c}$$

$$\mathbf{p}\times\tilde{\mathbf{F}}_+-\tilde{\mathbf{h}}_+\mathbf{p}+i\tilde{\mathbf{f}}_+\times\mathbf{t} = \mathbf{0}, \tag{22d}$$

$$\mathbf{p}\times\tilde{\mathbf{G}}_-+\tilde{\mathbf{j}}_-\cdot\mathbf{t} = \mathbf{0}, \tag{22e}$$

$$\mathbf{p}\times\tilde{\mathbf{J}}_--\tilde{\mathbf{g}}_-\mathbf{t}+i\tilde{\mathbf{j}}_-\times\mathbf{p} = \mathbf{0}, \tag{22f}$$

$$\mathbf{p}\times\tilde{\mathbf{H}}_++\tilde{\mathbf{f}}_+\cdot\mathbf{p} = \mathbf{0}, \tag{22g}$$

$$\mathbf{p}\times\tilde{\mathbf{f}}_+-i\tilde{\mathbf{F}}_+\times\mathbf{t}-\tilde{\mathbf{H}}_+\mathbf{p}+2m\tilde{\mathbf{j}}_- = \frac{2}{p^2}\mathbf{1}\times\mathbf{p}. \tag{22h}$$

We may take all the coefficient tensors to be transverse to $\mathbf{p}$ in the first index,

$$\mathbf{p}\cdot\tilde{\mathbf{f}}_+=\mathbf{0},\quad \mathbf{p}\cdot\tilde{\mathbf{F}}_+=\mathbf{0},\quad \mathbf{p}\cdot\tilde{\mathbf{g}}_-=0, \tag{23}$$

and so on, which is consistent with the transversality of the electric and magnetic fields appearing in the construction (18) of Q_1. This property then allows us to solve Eqs. (22d), (22e), (22f) and (22g) for $\tilde{\mathbf{F}}_+$, $\tilde{\mathbf{G}}_-$, $\tilde{\mathbf{H}}_+$, and $\tilde{\mathbf{J}}_-$ in terms of $\tilde{\mathbf{f}}_+$, $\tilde{\mathbf{g}}_-$, $\tilde{\mathbf{h}}_+$, and $\tilde{\mathbf{j}}_-$:

$$\begin{aligned}
\tilde{\mathbf{F}}_+ &= \frac{1}{p^2}\left(-\mathbf{p}\times\tilde{\mathbf{h}}_+\mathbf{p}+i\mathbf{p}\times\tilde{\mathbf{f}}_+\times\mathbf{t}\right), && (24a)\\
\tilde{\mathbf{G}}_- &= \frac{1}{p^2}\mathbf{p}\times\tilde{\mathbf{j}}_-\cdot\mathbf{t}, && (24b)\\
\tilde{\mathbf{J}}_- &= -\frac{1}{p^2}\left(\mathbf{p}\times\tilde{\mathbf{g}}_-\mathbf{t}-i\mathbf{p}\times\tilde{\mathbf{j}}_-\times\mathbf{p}\right), && (24c)\\
\tilde{\mathbf{H}}_+ &= \frac{1}{p^2}\mathbf{p}\times\tilde{\mathbf{f}}_+\cdot\mathbf{p}. && (24d)
\end{aligned}$$

The remaining four equations then imply that

$$\begin{aligned}
\mathbf{p}\times\tilde{\mathbf{g}}_-\left(p^2-t^2\right)+i\mathbf{p}\times\tilde{\mathbf{j}}_-\cdot(\mathbf{p}\times\mathbf{t})-2mp^2\tilde{\mathbf{h}}_+ &= \mathbf{0}, && (25a)\\
i\mathbf{p}\times\tilde{\mathbf{f}}_+\cdot(\mathbf{p}\times\mathbf{t})-2mp^2\tilde{\mathbf{g}}_- &= \mathbf{0}, && (25b)\\
\mathbf{p}\times\tilde{\mathbf{j}}_-\cdot(\mathbf{pp}-\mathbf{tt})-i\mathbf{p}\times\tilde{\mathbf{g}}_-\mathbf{p}\times\mathbf{t}-2mp^2\tilde{\mathbf{f}}_+ &= \mathbf{0}, && (25c)\\
\mathbf{p}\times\tilde{\mathbf{f}}_+\cdot\left[(\mathbf{tt}-\mathbf{1}t^2)-(\mathbf{pp}-\mathbf{1}p^2)\right]+i\mathbf{p}\times\tilde{\mathbf{h}}_+\mathbf{p}\times\mathbf{t}+2mp^2\tilde{\mathbf{j}}_- &= 2(\mathbf{1}\times\mathbf{p}). && (25d)
\end{aligned}$$

Equations (25b) and (25a) allow us to solve immediately for $\tilde{\mathbf{g}}_-$ and $\tilde{\mathbf{f}}_+$ in terms of $\tilde{\mathbf{j}}_-$ and $\tilde{\mathbf{h}}_+$:

$$\begin{aligned}
\tilde{\mathbf{g}}_- &= \frac{1}{2mp^2}i\mathbf{p}\times\tilde{\mathbf{f}}_+\cdot(\mathbf{p}\times\mathbf{t}), && (26a)\\
\tilde{\mathbf{h}}_+ &= \frac{i}{2mp^2}\left[\mathbf{p}\times\tilde{\mathbf{j}}_-\cdot(\mathbf{p}\times\mathbf{t})+(t^2-p^2)\frac{1}{2m}\tilde{\mathbf{f}}_+\cdot(\mathbf{p}\times\mathbf{t})\right], && (26b)
\end{aligned}$$

and then from Eqs. (25c) and (25d) we obtain two equations for $\tilde{\mathbf{j}}_-$ and $\tilde{\mathbf{f}}_+$:

$$\begin{aligned}
\mathbf{p}\times\tilde{\mathbf{j}}_-\cdot(\mathbf{tt}-\mathbf{pp})+2mp^2\tilde{\mathbf{f}}_+\cdot\left[\mathbf{1}+\frac{(\mathbf{p}\times\mathbf{t})(\mathbf{p}\times\mathbf{t})}{4m^2p^2}\right] &= \mathbf{0}, && (27a)\\
\mathbf{p}\times\tilde{\mathbf{f}}_+\cdot\left[(\mathbf{tt}-\mathbf{1}t^2)-(\mathbf{pp}-\mathbf{1}p^2)+\frac{t^2-p^2}{4m^2p^2}(\mathbf{p}\times\mathbf{t})(\mathbf{p}\times\mathbf{t})\right]&\\
+2mp^2\tilde{\mathbf{j}}_-\cdot\left[\mathbf{1}+\frac{(\mathbf{p}\times\mathbf{t})(\mathbf{p}\times\mathbf{t})}{4m^2p^2}\right] &= 2(\mathbf{1}\times\mathbf{p}). && (27b)
\end{aligned}$$

From Eq. (27a) we see that

$$\tilde{\mathbf{f}}_+\cdot(\mathbf{t}\times\mathbf{p})=\mathbf{0}. \tag{28}$$

Then we can solve Eq. (27a) for $\mathbf{f}_+$ in terms of $\mathbf{j}_-$, which when substituted into Eq. (27b) yields an equation that can be solved easily for $\mathbf{j}_-$.

In this way it is straightforward to solve for all the coefficient tensors. In terms of the denominator

$$\Delta = 4m^2p^2 + k^2, \tag{29}$$

where $\mathbf{k} = \mathbf{p} \times \mathbf{t}$, the nonzero tensor coefficients in Q_1 are

$$\begin{aligned}
\tilde{\mathbf{F}}_+ &= \frac{2i}{p^2\Delta}\mathbf{p} \times \mathbf{k}\,\mathbf{p}, && (30a)\\
\tilde{\mathbf{f}}_+ &= -\frac{2}{p^2\Delta}\mathbf{p} \times \mathbf{k}\,\mathbf{t}, && (30b)\\
\tilde{\mathbf{j}}_- &= \frac{4m}{\Delta}\mathbf{1} \times \mathbf{p}, && (30c)\\
\tilde{\mathbf{J}}_- &= -i\mathbf{j}_-, && (30d)\\
\tilde{\mathbf{h}}_+ &= -\frac{2i}{\Delta}\mathbf{k}, && (30e)\\
\tilde{\mathbf{H}}_+ &= 2\frac{\mathbf{p} \cdot \mathbf{t}}{p^2}\frac{\mathbf{k}}{\Delta}, && (30f)\\
\tilde{\mathbf{g}}_- &= \mathbf{0}, && (30g)\\
\tilde{\mathbf{G}}_- &= \frac{4m}{p^2\Delta}\mathbf{p} \times \mathbf{k}. && (30h)
\end{aligned}$$

Note that the parity constraint (19) is satisfied because the + quantities are even under $\mathbf{p} \to -\mathbf{p}$, $\mathbf{t} \to -\mathbf{t}$, while the $-$ quantities are odd. The time-reversal constraint (12b) is satisfied because of the presence of i in $\tilde{\mathbf{F}}_+$, $\tilde{\mathbf{J}}_-$, and $\tilde{\mathbf{h}}_+$, owing to $\mathcal{T}$ being an antiunitary operator. The odd functions undergo another sign change under $\mathcal{T}$ because all momenta change sign [see Eq. (21)].

IV. CONCLUSIONS

By constructing the first-order term in the Q operator and thus the leading approximation to the $\mathcal{C}$ operator, we have provided convincing evidence that the $\mathcal{PT}$-symmetric quantum electrodynamics originally proposed in Ref. [12] is unitary and that this construction enables us to obtain a unitary S matrix for the theory. Therefore, there can be little doubt that such a $\mathcal{PT}$-symmetric theory is self-consistent and one should now investigate whether such a theory may be used to describe natural phenomena. Indeed, this theory provides an interesting test of Gell-Mann's *Totalitarian Principle*, which states that "Everything which is not forbidden is compulsory" [15].

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