Physics 5583. Electrodynamics II.
Final Examination
Spring 2007

May 8, 2007

Instructions: This examination consists of three problems. If you get stuck on one part, assume a result and proceed onward. Do not hesitate to ask questions. GOOD LUCK!

1. In this problem we consider an infinitely-long hollow cylindrical wave-guide of square cross section, of side a. Recall that for single E modes, the electric and magnetic fields are given in terms of the mode functions ϕ and the voltage and current functions V and I, respectively, by ($\zeta = \sqrt{\mu/\epsilon}$)

$$\begin{aligned}
\mathbf{E}_\perp &= -\boldsymbol{\nabla}_\perp \phi(x,y) V(z), \\
\mathbf{H}_\perp &= -\hat{\mathbf{z}} \times \boldsymbol{\nabla}_\perp \phi(x,y) I(z), \\
E_z &= i\zeta \frac{\gamma^2}{k} \phi(x,y) I(z), \\
H_z &= 0,
\end{aligned}$$

and for a single H mode, the fields are given in terms of mode functions ψ ($\eta = \sqrt{\epsilon/\mu}$):

$$\begin{aligned}
\mathbf{E}_\perp &= \hat{\mathbf{z}} \times \boldsymbol{\nabla}_\perp \psi(x,y) V(z), \\
\mathbf{H}_\perp &= -\boldsymbol{\nabla}_\perp \psi(x,y) I(z), \\
H_z &= i\eta \frac{\gamma^2}{k} \psi(x,y) V(z), \\
E_z &= 0.
\end{aligned}$$

Here, for E modes the mode functions satisfy the eigenvalue equation

$$(\nabla_\perp + \gamma^2)\phi = 0,$$

with the boundary condition $\phi = 0$ on the boundary, except for $\gamma = 0$, where $\partial\phi/\partial s = 0$, $\hat{\mathbf{s}}$ being the tangent to the circumference. (The latter is called a T mode.) For the H modes, ψ satisfies the same eigenvalue condition, but the boundary condition is $\partial\psi/\partial n = 0$, that is, the derivative normal to the boundary vanishes.

(a) Determine the lowest E-mode eigenvalue and the corresponding eigenfunction. Do not worry about the normalization of the latter. Is there a T mode in this situation?

(b) Determine the lowest H-mode eigenvalue and the corresponding (unnormalized) eigenfunction.

(c) Which of the modes corresponds to the largest wavelength? For what wavelength range is this mode the only allowed (propagating) mode for the waveguide?

(d) Consider now the transmission-line equations satisfied by I and V

$$\begin{aligned}
\text{E-mode}: \quad & \frac{dI}{dz} = ik\eta V, \\
& \frac{dV}{dz} = ik\zeta\left(1 - \frac{\gamma^2}{k^2}\right) I, \\
\text{H-mode}: \quad & \frac{dI}{dz} = ik\eta\left(1 - \frac{\gamma^2}{k^2}\right) V, \\
& \frac{dV}{dz} = ik\zeta I.
\end{aligned}$$

Assuming the propagating form

$$I(z) = Ie^{i\kappa z}, \quad V(z) = Ve^{i\kappa z},$$

for the lowest mode found in part 1c, use these transmission-line equations to determine the guide wavelength $\lambda_g = 2\pi/\kappa$ in terms of the intrinsic wavelength $\lambda = 2\pi/k$ and the side of the guide a. What happens when the intrinsic wavelength equals the longest wavelength allowed in the guide?

(e) What is the impedance of the line, $Z = V/I$, in terms of λ_g and λ, or in terms of λ and a.

2. This problem demonstrates that classical electrodynamics cannot be applied to the Bohr model of the atom. Consider a model of the hydrogen atom in which the electron (charge $-e$ and mass m) orbits the nucleus (of charge $+e$ and infinite mass) in a circular orbit of radius r. Because of radiation losses, described by the dipole formula (Larmor formula) in Heaviside-Lorentz units

$$P = \frac{1}{6\pi}\frac{\ddot{\mathbf{d}}^2}{c^3},$$

the radius of the orbit decreases with time. Assume that the decrease is very small over a single orbital period.

(a) Using the relation between the kinetic and potential energy for a charged particle in a Coulomb field in a circular orbit, obtain a differential equation for the rate of change of the radius of the orbit in terms of the radius of the orbit and the constants given.

(b) Determine the time T required for the radius to decrease to zero (the nuclear size is very small), assuming that the initial radius is the Bohr radius,

$$r_B = \frac{\hbar}{\alpha m c},$$

where $\alpha = 1/137$ is the fine structure constant,

$$\alpha = \frac{e^2}{4\pi\hbar c}.$$

Express T as a function of α, $c = 3 \times 10^8$ m/s, and the Compton wavelength of the electron,

$$\lambda_c = \frac{\hbar}{mc} = 3.86 \times 10^{-13}\,\text{m}.$$

(c) Putting in the numbers, give the order of magnitude of the lifetime of the atom estimated in this way. Is this consistent with observation?

3. A light wave of definite frequency ω and wavevector $\mathbf{k}$ is characterized by the 4-vector propagation vector

$$k^\mu = (\omega/c, \mathbf{k}).$$

The invariant (proper) length-squared of this vector is $k_\mu k^\mu = 0$.

(a) Consider a wave propagating in a direction making an angle θ with respect to the z axis. Viewed by another observer moving with velocity $+v$ along the z axis, what is the frequency ω', in terms of ω, v, and θ (this is the Doppler effect), and

(b) what is the angle of propagation of the wave θ', in terms of θ and v (this is aberration)? Obtain these two results by noting that, like $x^\mu = (ct, \mathbf{x})$, k^μ transforms as a 4-vector under a Lorentz transformation.