Physics 5583. Electrodynamics II.
Midterm Examination II
Spring 2007

April 13, 2007

Instructions: This examination consists of three problems. If you get stuck on one part, assume a result and proceed onward. Do not hesitate to ask questions. GOOD LUCK!

1. This problem derives a general formula for the power radiated into a given element of solid angle.

   (a) Recall that in the radiation gauge the vector potential is given by

   $$\mathbf{A}(\mathbf{r}) = \int (d\mathbf{r}') \frac{\frac{1}{c}\mathbf{j}(\mathbf{r}', t_r)}{4\pi|\mathbf{r}-\mathbf{r}'|}, \quad t_r = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}.$$

   From this evaluate the magnetic field $\mathbf{B}$ in the radiation zone, $r \gg r'$, if the origin is in the finite region of the source current. Derive and use the relation

   $$\boldsymbol{\nabla} f(t_r) = -\frac{1}{c}\hat{\mathbf{n}}\frac{\partial}{\partial t} f(t_r).$$

   (What is $\hat{\mathbf{n}}$ and what is the radiation-zone version of t_r?)

   (b) Similarly, give an expression for the electric field $\mathbf{E}$ in the radiation zone, and relate it to the corresponding magnetic field. Hint: Prove that

   $$\frac{\partial}{\partial t}\rho(\mathbf{r}', t_r) + \boldsymbol{\nabla}' \cdot \mathbf{j}(\mathbf{r}', t_r) = \frac{\hat{\mathbf{n}}}{c} \cdot \frac{\partial}{\partial t}\mathbf{j}(\mathbf{r}', t_r),$$

where the $\boldsymbol{\nabla}'$ differentiates both the explicit $\mathbf{r}'$ and the $\mathbf{r}'$ dependence of t_r.

(c) Show that the Poynting vector can be written as

$$\mathbf{S} = c\hat{\mathbf{n}}|\mathbf{B}|^2.$$

(d) From this, show that the power radiated into solid angle $d\Omega$ is

$$\frac{dP}{d\Omega} = \frac{1}{(4\pi)^2 c^3}\left[\hat{\mathbf{n}} \times \int (d\mathbf{r}')\frac{\partial}{\partial t}\mathbf{j}(\mathbf{r}', t_r)\right]^2.$$

2. In this problem, we use the result of the previous problem to calculate the power radiated by an antenna.

(a) Let us model, rather unrealistically, the antenna by a wire of length l, of negligible cross section, carrying a current flowing in the z direction, with density

$$j_z = I\delta(x)\delta(y)\sin\omega t, \quad -\frac{l}{2} \le z \le \frac{l}{2}.$$

Show that in terms of

$$\lambda = \frac{2\pi c}{\omega},$$

the angular distribution of the power radiated, averaged over one cycle of oscillation, is

$$\frac{dP}{d\Omega} = \frac{I^2}{8\pi^2 c}\frac{\sin^2\theta\sin^2\left(\frac{\pi l}{\lambda}\cos\theta\right)}{\cos^2\theta}.$$

(b) Show that the result of the dipole approximation holds in the limit of a very short antenna, $l \ll \lambda$. [Hint: Use local conservation of charge to calculate the electric dipole moment.]

(c) Qualitatively sketch the radiation pattern for $l = \frac{3}{2}\lambda$. In particular, at what angles are the maxima and minima of the radiated power?

3. Recall that the electromagnetic stress tensor is

$$T^{\mu\nu} = F^{\mu\lambda}F^{\nu}{}_{\lambda} - g^{\mu\nu}\frac{1}{4}F^{\kappa\lambda}F_{\kappa\lambda}.$$

(a) Show that $T^{\mu\nu} = T^{\nu\mu}$ and that $T^\lambda{}_\lambda = 0$.

(b) Prove, directly from Maxwell's equations

$$\partial_\nu F^{\mu\nu} = \frac{1}{c} j^\mu, \quad \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0,$$

that

$$\partial_\nu T^{\mu\nu} = -F^{\mu\nu} \frac{1}{c} j_\nu.$$

(c) Show that the particle stress tensor,

$$t^{\mu\nu}(x) = \int_{-\infty}^{\infty} d\tau \, m_0 c \frac{dx^\mu(\tau)}{d\tau} \frac{dx^\nu(\tau)}{d\tau} \delta^{(4)}(x - x(\tau)),$$

where $x^\mu(\tau)$ is the space-time trajectory of the particle, which obeys the equation of motion

$$m_0 \frac{d^2 x^\mu(\tau)}{d\tau^2} = \frac{e}{c} F^{\mu\nu}(x(\tau)) \frac{dx_\nu(\tau)}{d\tau},$$

satisfies

$$\partial_\nu t^{\mu\nu} = \frac{1}{c} F^{\mu\nu} j_\nu.$$

Here the electric current is given by a similar proper-time integral,

$$\frac{1}{c} j^\mu(x) = \int_{-\infty}^{\infty} d\tau \, e \frac{dx^\mu}{d\tau} \delta^{(4)}(x - x(\tau)).$$

Thus the divergence of $T^{\mu\nu} + t^{\mu\nu}$ is zero, which expresses the balance of energy and momentum between the particle and the field.