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Relativistic Transformations

3.1 Four-Dimensional Notation

A space-time coordinate can be represented by a contravariant vector,

$$x^\mu \ : \quad x^0 = ct \ , \quad x^1 = x \ , \quad x^2 = y \ , \quad x^3 = z \ , \tag{3.1}$$

where μ is an index which takes on the values 0, 1, 2, 3. The corresponding covariant vector is

$$x_\mu \ : \quad x_0 = -ct \ , \quad x_1 = x \ , \quad x_2 = y \ , \quad x_3 = z \ . \tag{3.2}$$

The contravariant and covariant vector components are related by the metric tensor $g_{\mu\nu}$,

$$x_\mu = g_{\mu\nu} x^\nu \ , \tag{3.3}$$

which uses the Einstein summation convention of summing over repeated covariant and contravariant indices,

$$g_{\mu\nu} x^\nu = \sum_{\nu=0}^{3} g_{\mu\nu} x^\nu \ . \tag{3.4}$$

From the above explicit forms for x_μ and x^ν we read off, in matrix form (here the first index labels the rows, the second the columns, both enumerated from 0 to 3)[1]

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \tag{3.5}$$

[1] In this book, we use what could be referred to as the democratic metric (formerly the East-coast metric), in which the signature is dictated by the larger number of entries.

where evidently $g_{\mu\nu}$ is symmetric,

$$g_{\mu\nu} = g_{\nu\mu} \, . \tag{3.6}$$

Similarly,

$$x^\mu = g^{\mu\nu} x_\nu \, , \tag{3.7}$$

where

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \, , \quad g^{\mu\nu} = g^{\nu\mu} \, . \tag{3.8}$$

The four-dimensional analogue of a rotationally invariant length is the proper length s or the proper time τ:

$$s^2 = -c^2\tau^2 = x^\mu x_\mu = x^\mu g_{\mu\nu} x^\nu = x_\mu g^{\mu\nu} x_\nu = \mathbf{r}\cdot\mathbf{r} - (ct)^2 \, . \tag{3.9}$$

Recall the transformation of a scalar field under a coordinate displacement,[2] as in (1.76),

$$\delta\phi(\mathbf{r}) = -\delta\mathbf{r}\cdot\boldsymbol{\nabla}\phi(\mathbf{r}), \quad \boldsymbol{\nabla} = \frac{\partial}{\partial\mathbf{r}} \, . \tag{3.11}$$

The corresponding four-dimensional statement is

$$\delta\phi(x) = -\delta x^\mu \partial_\mu \phi(x) \quad \partial_\mu = \frac{\partial}{\partial x^\mu} \, , \tag{3.12}$$

which shows the definition of the covariant gradient operator, so defined in order that $\partial_\mu x^\mu$ be invariant. The corresponding contravariant gradient is

$$\partial^\mu = g^{\mu\nu}\partial_\nu = \frac{\partial}{\partial x_\mu} \, . \tag{3.13}$$

Using these operators we can write the equation of electric current conservation, (1.14),

$$\boldsymbol{\nabla}\cdot\mathbf{j} + \frac{\partial}{\partial t}\rho = 0 \, , \tag{3.14}$$

in the four-dimensional form

$$\partial_\mu j^\mu = 0 \, , \tag{3.15}$$

[2] There is a sign change relative to what appears in Chap. 1. That is because we are now considering passive transformations. Thus, under an infinitesimal coordinate displacement, a scalar field transforms according to $\overline{\phi}(x+\delta x) = \phi(x)$, while $\delta\phi$ is defined at the same coordinate:

$$\delta\phi(x) = \overline{\phi}(x) - \phi(x) = \phi(x - \delta x) - \phi(x) \, . \tag{3.10}$$

where we define the components of the electric current four-vector as

$$j^\mu \ : \quad j^0 = c\rho \ , \quad \{j^i\} = \mathbf{j} \ , \tag{3.16}$$

where we have adopted the convention that Latin indices run over the values 1, 2, 3, corresponding to the three spatial directions. Note that (3.16) is quite analogous to the construction of the position four-vector, (3.1).

The invariant interaction term (1.61)

$$\mathcal{L}_{\text{int}} = -\rho\phi + \frac{1}{c}\mathbf{j} \cdot \mathbf{A} \tag{3.17}$$

has the four-dimensional form

$$\frac{1}{c} j^\mu A_\mu \ , \tag{3.18}$$

where

$$A_\mu = g_{\mu\nu} A^\nu \ , \quad A^0 = \phi \ , \quad \{A^i\} = \mathbf{A} \ . \tag{3.19}$$

The four dimensional generalization of

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{3.20}$$

is the tensor construction

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \tag{3.21}$$

where the antisymmetric field strength tensor

$$F_{\mu\nu} = -F_{\nu\mu} \tag{3.22}$$

contains the magnetic field components as

$$F_{23} = B_1 \ , \quad F_{31} = B_2 \ , \quad F_{12} = B_3 \ , \tag{3.23}$$

which may be presented more succinctly as

$$F_{ij} = \epsilon_{ijk} B^k \ , \tag{3.24}$$

which uses the totally antisymmetric Levi-Cività symbol:

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = -\epsilon_{213} = -\epsilon_{132} = -\epsilon_{321} = 1 \ , \tag{3.25}$$

all other components being zero. The construction (3.21) includes the other potential statement (1.48),

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A} \ , \tag{3.26}$$

provided

$$F_{0i} = -E_i \; . \tag{3.27}$$

Alternatively, with

$$F^{\mu}{}_{\nu} = g^{\mu\lambda} F_{\lambda\nu} \; , \tag{3.28}$$

we have

$$F^{0}{}_{i} = E_i \; . \tag{3.29}$$

Maxwell's equations with only electric currents present are summarized by

$$\partial_\nu F^{\mu\nu} = \frac{1}{c} j^\mu \; , \tag{3.30a}$$

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \; , \tag{3.30b}$$

where

$$F^{\mu\nu} = F^{\mu}{}_{\lambda} g^{\lambda\nu} = g^{\mu\kappa} F_{\kappa\lambda} g^{\lambda\nu} \; . \tag{3.31}$$

It is convenient to define a dual field-strength tensor by

$$^{*}F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda} F_{\kappa\lambda} = -{}^{*}F^{\nu\mu} \; , \tag{3.32}$$

where $\epsilon^{\mu\nu\kappa\lambda}$ is the four-dimensional totally antisymmetric Levi-Cività symbol, which therefore vanishes if any two of the indices are equal, normalized by

$$\epsilon^{0123} = +1 \; . \tag{3.33}$$

We now have

$$^{*}F^{01} = F_{23} = B_1 \; , \quad {}^{*}F^{02} = F_{31} = B_2 \; , {}^{*}F^{03} = F_{12} = B_3 \; , \tag{3.34}$$

and

$$^{*}F^{23} = F_{01} = -E_1 \; , \quad {}^{*}F^{31} = F_{02} = -E_2 \; , {}^{*}F^{12} = F_{03} = -E_3 \; , \tag{3.35}$$

so indeed the dual transformation corresponds to the replacement

$$\mathbf{E} \to \mathbf{B} \; , \quad \mathbf{B} \to -\mathbf{E} \; . \tag{3.36}$$

[This is a special case of the duality rotation (1.219).] Note that two dual operations brings you back to the beginning:

$$^{*}({}^{*}F^{\mu\nu}) = -F^{\mu\nu} \; . \tag{3.37}$$

Using the dual, Maxwell's equations including both the electric (j^μ) and the magnetic ($^{*}j^\mu$) currents (called j_e and j_m in the problems in the previous chapter) are given by

$$\partial_\nu F^{\mu\nu} = \frac{1}{c} j^\mu \; , \quad \partial_\nu {}^{*}F^{\mu\nu} = \frac{1}{c} {}^{*}j^\mu \; , \tag{3.38}$$

where both currents must be conserved,

$$\partial_\mu j^\mu = c\partial_\mu \partial_\nu F^{\mu\nu} = 0 \,, \tag{3.39a}$$

$$\partial_\mu {}^* j^\mu = c\partial_\mu \partial_\nu {}^* F^{\mu\nu} = 0 \,, \tag{3.39b}$$

because of the symmetry in μ and ν of $\partial_\mu \partial_\nu$ and the antisymmetry of $F^{\mu\nu}$ and ${}^*F^{\mu\nu}$.

We had earlier in (3.9) introduced the proper time. The corresponding differential statement is

$$\mathrm{d}\tau = \frac{1}{c}\sqrt{-\mathrm{d}x^\mu \mathrm{d}x_\mu} = \mathrm{d}t\sqrt{1-\frac{v^2}{c^2}} \,, \tag{3.40}$$

which is an invariant time interval. The particle equations of motion using τ as the time parameter read (see Problem 3.1)

$$m_0 \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = \frac{e}{c} F^\mu{}_\nu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \,, \tag{3.41}$$

to which is to be added $\frac{g}{c}{}^*F^\mu{}_\nu \mathrm{d}x^\nu/\mathrm{d}\tau$ if the particle possesses magnetic charge g. We can write down three alternative forms for the action of the particle:

$$W_{12} = \int_2^1 \left(-m_0 c^2 \mathrm{d}\tau + \frac{e}{c} A_\mu \mathrm{d}x^\mu\right) \tag{3.42a}$$

$$= \int_2^1 \mathrm{d}\tau \left[\frac{1}{2} m_0 \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x_\mu}{\mathrm{d}\tau} - c^2\right) + \frac{e}{c} A_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right] \tag{3.42b}$$

$$= \int_2^1 \mathrm{d}\tau \left[p_\mu \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} - v^\mu\right) + \frac{1}{2} m_0 \left(v^\mu v_\mu - c^2\right) + \frac{e}{c} A_\mu v^\mu\right] . \tag{3.42c}$$

In the last two forms, τ is an independent parameter, with the added requirement that each generator G [recall the action principle states $\delta W_{12} = G_1 - G_2$] is independent of $\delta\tau$. In the third version, where x^μ, v^μ, and p^μ are independent dynamical variables, it is a consequence of the action principle that

$$v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \,, \quad p^\mu = m_0 v^\mu + \frac{e}{c} A^\mu \,, \quad v^\mu v_\mu = -c^2 \,, \tag{3.43a}$$

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = \frac{e}{c} \partial_\mu A_\lambda v^\lambda \,. \tag{3.43b}$$

The invariant Lagrange function for the electromagnetic field (1.60) is

$$\mathcal{L}_f = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{E^2 - B^2}{2} \,. \tag{3.44}$$

The energy-momentum, or stress tensor, subsumes the energy density, the momentum density (or energy flux vector) and the three-dimensional stress tensor:

$$T^{\mu\nu} = T^{\nu\mu} = F^{\mu\lambda}F^{\nu}{}_{\lambda} - g^{\mu\nu}\frac{1}{4}F^{\kappa\lambda}F_{\kappa\lambda} \; ; \tag{3.45}$$

It has the property of being traceless:

$$T^{\mu}{}_{\mu} = g_{\mu\nu}T^{\mu\nu} = 0 \; , \tag{3.46}$$

and has the following explicit components:

$$T^{00} = U \; , \quad T^{0}{}_{k} = \frac{1}{c}S_k = cG_k \; , \quad T_{ij} = \mathsf{T}_{ij} \; , \tag{3.47}$$

in terms of the energy density, (1.20a), the energy flux vector (1.20b) or momentum density (1.20c), and the stress tensor (1.20d). It satisfies the equation

$$\partial_\nu T^{\mu\nu} = -F^{\mu\nu}\frac{1}{c}j_\nu \; , \tag{3.48}$$

which restates the energy and momentum conservation laws (1.44a) and (1.44b).

3.2 Field Transformations

A Lorentz transformation, or more properly a *boost,* is a transformation that mixes the time and space coordinates without changing the invariant distance s^2. An infinitesimal transformation of this class is

$$\delta\mathbf{r} = \delta\mathbf{v}t \; , \quad \delta t = \frac{1}{c^2}\delta\mathbf{v}\cdot\mathbf{r} \; , \tag{3.49}$$

where $-\delta\mathbf{v}$ is the velocity with which the new coordinate frame moves relative to the old one. (It is assumed that the two coordinate frames coincide at $t = 0$.) In terms of the four-vector position, $x^\mu = (ct, \mathbf{r})$, we can write this result compactly as

$$\delta x^\mu = \delta\omega^{\mu\nu}x_\nu \; , \tag{3.50}$$

where the only nonzero components of the transformation parameter $\delta\omega^{\mu\nu}$ are

$$\delta\omega^{0i} = -\delta\omega^{i0} = \frac{\delta v^i}{c} \; . \tag{3.51}$$

Ordinary rotations of course also preserve s^2, so they must be included in the transformations (3.50), and they are, corresponding to $\delta\omega^{\mu\nu}$ having no time components, and spatial components

$$\delta\omega^{ij} = -\epsilon^{ijk}\delta\omega_k \; , \tag{3.52}$$

so, as in (1.81),

$$\delta\mathbf{r} = \delta\boldsymbol{\omega}\times\mathbf{r} \; . \tag{3.53}$$

In fact, the only property $\delta\omega^{\mu\nu}$ must have in order to preserve the invariant length s^2 is antisymmetry:

$$\delta\omega^{\mu\nu} = -\delta\omega^{\nu\mu} , \tag{3.54}$$

for

$$\delta(x^\mu x_\mu) = 2\delta\omega^{\mu\nu} x_\mu x_\nu = 0 , \tag{3.55}$$

and a scalar product, such as that in $j^\mu A_\mu$ is similarly invariant. Any infinitesimal transformation with this property we will dub a Lorentz transformation.

Now consider the transformation of a four-vector field, such as the vector potential, $A^\mu = (\phi, \mathbf{A})$. This field undergoes the same transformation as given by the coordinate four-vector, but one must also transform to the new coordinate representing the same physical point. That is, under a Lorentz transformation,

$$A^\mu(x) \to \overline{A}^\mu(\overline{x}) = A^\mu(x) + \delta\omega^{\mu\nu} A_\nu(x) , \tag{3.56}$$

where

$$\overline{x}^\mu = x^\mu + \delta x^\mu = x^\mu + \delta\omega^{\mu\nu} x_\nu . \tag{3.57}$$

So that the transformation may be considered a field variation only, we define the change in the field at the same coordinate value (which refers to different physical points in the two frames):

$$\begin{aligned} \delta A^\mu(x) &= \overline{A}^\mu(x) - A^\mu(x) \\ &= A^\mu(x - \delta x) + \delta\omega^{\mu\nu} A_\nu(x) - A^\mu(x) \\ &= -\delta x^\nu \partial_\nu A^\mu(x) + \delta\omega^{\mu\nu} A_\nu(x) . \end{aligned} \tag{3.58}$$

The four-vector current $j^\mu = (c\rho, \mathbf{j})$ must transform in the same way:

$$\delta j^\mu = -\delta x^\nu \partial_\nu j^\mu(x) + \delta\omega^{\mu\nu} j_\nu(x) . \tag{3.59}$$

A scalar field, $\lambda(x)$, on the other hand only undergoes the coordinate transformation:

$$\lambda(x) \to \overline{\lambda}(\overline{x}) = \lambda(x) , \tag{3.60}$$

so

$$\delta\lambda(x) = -\delta x^\nu \partial_\nu \lambda(x) . \tag{3.61}$$

Because a vector potential can be changed by a gauge transformation,

$$A^\mu \to A^\mu + \partial^\mu \lambda , \tag{3.62}$$

without altering any physical quantity, in particular the field strength tensor $F^{\mu\nu}$, the transformation law for the vector potential must follow by differentiating that of λ, and indeed it does.

What about the transformation property of the field strength tensor? Again, it follows by direct differentiation:

$$\begin{aligned}\delta F_{\mu\nu} &= \delta(\partial_\mu A_\nu - \partial_\nu A_\mu)\\ &= -\delta x^\lambda \partial_\lambda F_{\mu\nu} - (\partial_\mu \delta x^\lambda)\partial_\lambda A_\nu + (\partial_\nu \delta x^\lambda)\partial_\lambda A_\mu + \delta\omega_{\nu\lambda}\partial_\mu A^\lambda - \delta\omega_{\mu\lambda}\partial_\nu A^\lambda\\ &= -\delta x^\lambda \partial_\lambda F_{\mu\nu} + \delta\omega_\mu{}^\lambda F_{\lambda\nu} + \delta\omega_\nu{}^\lambda F_{\mu\lambda}\ . \end{aligned} \tag{3.63}$$

So we see that each index of a tensor transforms like that of a vector. From this it is easy to work out how the components of the electric and magnetic fields transform under a boost (3.51). Apart from the coordinate change – which just says we are evaluating fields at the same physical point – we see [cf. (1.230)]

$$‘\delta’\mathbf{E} = -\frac{\delta\mathbf{v}}{c} \times \mathbf{B}\ , \tag{3.64a}$$

$$‘\delta’\mathbf{B} = \frac{\delta\mathbf{v}}{c} \times \mathbf{E}\ , \tag{3.64b}$$

The proof of the Lorentz invariance the relativistic Lagrangian is now immediate. That is,

$$\delta\mathcal{L} = -\delta x^\lambda \partial_\lambda \mathcal{L}\ , \tag{3.65}$$

which just says that $\overline{\mathcal{L}}(\overline{x}) = \mathcal{L}(x)$, implying that $\delta W = \delta \int (dx)\mathcal{L}(x) = 0$. We have already remarked that ‘δ’$\mathcal{L}_{\text{int}} = 0$. The invariance of the field Lagrangian (3.44) is simply the statement

$$‘\delta’\mathcal{L}_f = \delta\omega_{\mu\nu} F^{\mu\lambda} F^\nu{}_\lambda = 0\ , \tag{3.66}$$

and the particle action in (3.42a)–(3.42c) is manifestly invariant.

3.3 Problems for Chap. 3

1. Show that the time and space components of (3.41) are equivalent to the equations of motion (1.45a) and (1.45b) provided the relativistic form of the particle kinetic energy and momentum, (1.18a) and (1.18b), are employed.
2. Derive the first form of the particle action (3.42a) from the relativistic particle Lagrangian $-m_0c^2\sqrt{1 - v^2/c^2}$ and the interaction (3.17).
3. Obtain the equations resulting from variations of the second form of the particle action (3.42b) with respect to both x^μ and τ variations, and verify that these are as expected.
4. A covariant form for the current vector of a moving point charge e is the proper-time integral

$$\frac{1}{c} j^\mu(x) = \int_{-\infty}^{\infty} \mathrm{d}\tau\, e\, \frac{\mathrm{d}x^\mu(\tau)}{\mathrm{d}\tau} \delta(x - x(\tau))\ . \tag{3.67}$$

Verify that

$$\partial_\mu j^\mu = 0\ , \tag{3.68}$$

under the assumption that the charge is infinitely remote at $\tau = \pm\infty$. Show that

$$\int (\mathrm{d}\mathbf{r}) \frac{1}{c} j^0(x) = e \,, \tag{3.69}$$

provided $\mathrm{d}x^0(\tau)/\mathrm{d}\tau$ is always positive. The stress tensor $T^{\mu\nu}$ for a mass point is given analogously by

$$T^{\mu\nu}(x) = \int_{-\infty}^{\infty} \mathrm{d}\tau \, m_0 c \frac{\mathrm{d}x^\mu(\tau)}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu(\tau)}{\mathrm{d}\tau} \delta(x - x(\tau)) \,. \tag{3.70}$$

Verify that

$$\partial_\nu T^{\mu\nu}(x) = 0 \,, \tag{3.71}$$

provided the particle is unaccelerated ($\mathrm{d}^2 x^\mu(\tau)/\mathrm{d}\tau^2 = 0$). Then show that

$$\int (\mathrm{d}\mathbf{r}) \, T^{0\nu} = m_0 c \frac{\mathrm{d}x^\nu(\tau)}{\mathrm{d}\tau} \,. \tag{3.72}$$

Does this comprise the expected values for the energy and momentum (multiplied by c) of a uniformly moving particle?

5. Suppose the particle of the previous problem is accelerated – it carries charge e and moves in an electromagnetic field. Use the covariant equations of motion (3.41) to show that

$$\partial_\nu T^{\mu\nu}_{\mathrm{part}} = \frac{1}{c} F^\mu{}_\nu j^\nu \,. \tag{3.73}$$

What do you conclude by comparison with the corresponding divergence of the electromagnetic stress tensor, (3.48)?

6. The next several problems refer to a purely electromagnetic model of the electron described first in [12]. A spherically symmetrical distribution of charge e at rest has the potentials $\phi = ef(r^2)$, $\mathbf{A} = \mathbf{0}$, where, at distances large compared with its size, $f(r^2) \sim 1/\sqrt{r^2}$. As observed in a frame in uniform relative motion, the potentials are

$$A^\mu(x) = \frac{e}{c} v^\mu f(\xi^2) \,, \quad \xi^\mu = x^\mu + \frac{v^\mu}{c} \left(\frac{v^\lambda}{c} x_\lambda \right) \,, \tag{3.74}$$

where

$$v^\lambda \xi_\lambda = 0 \,, \quad \xi^2 = x^2 + \left(\frac{v^\lambda}{c} x_\lambda \right)^2 \,. \tag{3.75}$$

Check that for motion along the z axis with velocity v,

$$\xi^2 = x^2 + y^2 + \frac{(z - vt)^2}{1 - v^2/c^2} \,, \tag{3.76}$$

as could be inferred from Problem 31.1 of [9]. Compute the field strengths $F^{\mu\nu}$ and evaluate the electromagnetic field stress tensor (3.45).

7. From the previous problem, use the field equation (3.30a) to produce $j^\mu(x)$. Check that $\partial_\mu j^\mu = 0$. Construct $F^{\mu\nu}\frac{1}{c}j_\nu$ and note that its vector nature lets one write
$$F^{\mu\nu}\frac{1}{c}j_\nu = -\partial^\mu t(\xi^2)\ . \tag{3.77}$$
Exhibit $t(\xi^2)$ for the example $f(\xi^2) = (\xi^2 + a^2)^{-1/2}$. Inasmuch as the field tensor obeys (3.48)
$$\partial_\nu T_f^{\mu\nu} = -F^{\mu\nu}\frac{1}{c}j_\nu = \partial^\mu t\ , \tag{3.78}$$
one has realized a divergenceless electromagnetic tensor:
$$T^{\mu\nu} = T_f^{\mu\nu} - g^{\mu\nu}t\ , \quad \partial_\nu T^{\mu\nu} = 0\ . \tag{3.79}$$
It is the basis of a purely electromagnetic relativistic model of mass. There is, however, an ambiguity, because from (3.75)
$$\partial_\nu\left(\frac{v^\mu}{c}\frac{v^\nu}{c}t(\xi^2)\right) = 0\ . \tag{3.80}$$
Therefore,
$$T^{\mu\nu} = T_f^{\mu\nu} - \left(g^{\mu\nu} + \frac{v^\mu}{c}\frac{v^\nu}{c}\right)t\ , \tag{3.81}$$
for example, is also a possible electromagnetic tensor. Choice (3.79) has the property that the momentum density of the moving system (multiplied by c) is just that of the field,
$$T^{0k} = T_f^{0k} = (\mathbf{E}\times\mathbf{B})_k\ . \tag{3.82}$$
Choice (3.81) is such that the energy density of the system at rest is just that of the field,
$$\mathbf{v} = \mathbf{0}: \quad T^{00} = T_f^{00} = \frac{E^2}{2}\ . \tag{3.83}$$
One cannot have both. That requires $t = 0$; that is, no charge. The system then is an electromagnetic pulse – it moves at the speed c.
8. Without specializing $f(\xi^2)$, integrate over all space (by introducing the variable $z' = (z - vt)/\sqrt{1 - v^2/c^2}$) to show that, whether one uses tensor (3.79) or (3.81),
$$E = \int(\mathrm{d}\mathbf{r})\,T^{00} = \frac{mc^2}{\sqrt{1 - v^2/c^2}}\ , \quad p_k = \frac{1}{c}\int(\mathrm{d}\mathbf{r})\,T^0{}_k = \frac{mv_k}{\sqrt{1 - v^2/c^2}}\ . \tag{3.84}$$
What numerical factor relates m is scheme (3.79) to that in scheme (3.81)?
9. Repeat the action discussion following from (3.42c) with $m_0 = 0$ and unspecified $f(\xi^2)$. What mass emerges?

10. Verify that the Maxwell equations involving magnetic currents, the second set in (3.38), can also be given by

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = \epsilon_{\mu\nu\lambda\kappa} \frac{1}{c} {}^* j^\kappa \ . \tag{3.85}$$

11. A particle has velocity components $v_x = \frac{\mathrm{d}x}{\mathrm{d}t}$ and $v_z = \frac{\mathrm{d}z}{\mathrm{d}t}$ in one coordinate frame. There is a second frame with relative velocity $\mathbf{v}$ along the z axis. What are the velocity components $v'_x = \frac{\mathrm{d}x'}{\mathrm{d}t'}$ and $v'_z = \frac{\mathrm{d}z'}{\mathrm{d}t'}$ in this frame? Give a simple interpretation of the v'_x result for $v_z = 0$.
12. Let the motion referred to in the previous problem be that of light, moving at angle θ with respect to the z axis. Find $\cos\theta'$ and $\sin\theta'$ in terms of $\cos\theta$ and $\sin\theta$. Check that $\cos^2\theta' + \sin^2\theta' = 1$. Exhibit θ' explicitly when $\beta = v/c \ll 1$.
13. The infinitesimal transformation contained in (3.51)

$$\delta\mathbf{p} = \frac{\delta\mathbf{v}}{c}\frac{E}{c} \ , \qquad \frac{\delta E}{c} = \frac{\delta\mathbf{v}}{c} \cdot \mathbf{p} \ , \tag{3.86}$$

identify the four-vector of momentum $p^\mu = (E/c, \mathbf{p})$ What is the value of the invariant $p^\mu p_\mu$ for a particle of rest mass m_0? Apply the analogue of the space-time transformation equations

$$t' = \frac{t + \mathbf{v}\cdot\mathbf{r}/c^2}{\sqrt{1 - v^2/c^2}} \ , \qquad \mathbf{v}\cdot\mathbf{r}' = \mathbf{v}\cdot\frac{\mathbf{r} + \mathbf{v}t}{\sqrt{1 - v^2/c^2}} \tag{3.87}$$

to find the energy and momentum of a moving particle from their values when the particle is at rest.
14. A body of mass M is at rest relative to one observer. Two photons, each of energy ϵ, moving in opposite directions along the x-axis, fall on the body, and are absorbed. Since the photons carry equal and opposite momenta, no net momentum is transferred to the body, and it remains at rest. Another observer is moving slowly along the y axis. Relative to him, the two photons and the body, both before and after the absorption act, have a common velocity $\mathbf{v}$ ($|\mathbf{v}| \ll c$) along the y axis, Reconcile conservation of the y-component of momentum with the fact that the velocity of the body does *not* change when the photons are absorbed.
15. Show, very simply, that $\mathbf{B}$, the magnetic field of a uniformly moving charge is $\frac{1}{c}\mathbf{v}\times\mathbf{E}$. Then consider two charges, moving with a common velocity $\mathbf{v}$ along parallel tracks, and show that the magnetic force between them is *opposite* to the electric force, and smaller by a factor of v^2/c^2. (This is an example of the rule that like *charges* repel, like *currents* attract.) Can you derive the same result by Lorentz transforming the equation of motion in the common rest frame of the two charges? (Hint: Coordinates perpendicular to the line of relative motion are unaffected by the transformation.)
16. This continues Problems 1.13–17. The relativistic formula

$$\mathrm{d}N = \frac{(\mathrm{d}\mathbf{k})}{k^0}\frac{\alpha}{4\pi^2}\left(\frac{v_1}{kv_1} - \frac{v_2}{kv_2}\right)^2 \tag{3.88}$$

describes the number of photons emitted with momentum k^μ under the deflection of a particle: $v_1^\mu \to v_2^\mu$. (Here, the scalar product is denoted by $kv_1 = k^\mu v_{1\mu}$, etc.) More realistic is a collision of two particles, with masses m_a, m_b and charges e_a and e_b. When, as a result of a collision in which the particle velocities change from v_{a2}, v_{b2} to v_{a1}, v_{b2}, what is $\mathrm{d}N$? Suppose this collision satisfies the conservation of energy-momentum, $(p_a^\mu + p_b^\mu)_1 = (p_a^\mu + p_b^\mu)_2$. Rewrite your expression for $\mathrm{d}N$ in terms of the p^μ rather than the v^μ. What follows if it should happen that e/kp has the same value for both particles, before and after the collision. Connect the nonrelativistic limit of this circumstance with Problem 1.13. Verify that this special circumstance does hold relativistically in the head-on, center-of-mass collision, where all momenta are of equal magnitude, for radiation *perpendicular* to the line of motion of the particles, provided $e_a/E_a = e_b/E_b$. What restriction does this impose on the energy if the particles are identical (same charge and rest mass)?

17. Use the fact that

$$k_\mu\left(\frac{v_1^\mu}{kv_1} - \frac{v_2^\mu}{kv_2}\right) = 0 \tag{3.89}$$

for example to show that

$$\left(\frac{v_1}{kv_1} - \frac{v_2}{kv_2}\right)^2 = \left(\mathbf{n}\times\left(\frac{\mathbf{v}_1}{kv_1} - \frac{\mathbf{v}_2}{kv_2}\right)\right)^2 . \tag{3.90}$$

Repeat the calculation of Problem 1.16 using this form and show the identity of the two results.

18. From the response of a particle momentum to an infinitesimal Lorentz transformation (3.86), find the infinitesimal change of the particle velocity $\mathbf{V}$ when $\mathbf{V}$ and $\delta\mathbf{v}$ are in the same direction. Compare your result with the implication of the formula for the relativistic addition of velocities.
19. Light travels at the speed c/n in a stationary, nondispersive medium. What is the speed of light when this medium is moving at speed v parallel or antiparallel to the direction of the light? To what does this simplify when $v/c \ll 1$?
20. An infinitesimal Lorentz transformation (boost) is characterized by a parameter $\delta\theta = \delta v/c$. Assuming that $\delta\mathbf{v}$ lies along the z direction, construct and solve the first-order differential equations obeyed by $ct(\theta) \pm z(\theta)$. What do the solutions tell you about the relation between θ and v/c? How does the addition of velocity formula read in terms of the corresponding θs? (The angle θ is often referred to as the "rapidity.")
21. The frequency ω and the propagation vector $\mathbf{k}$ of a plane wave form a four-vector: $k^\mu = (\omega/c, \mathbf{k})$. Check that $k^\mu k_\mu = 0$ and that $\exp(ik_\mu x^\mu) = \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$. Use Lorentz transformations to show that radiation, of

frequency ω, propagating at an angle θ with respect to the z axis, will, to an observer moving with relative velocity $v = \beta c$ along the z axis, have the frequency

$$\omega' = \frac{1}{\sqrt{1-\beta^2}}\omega(1-\beta\cos\theta) \tag{3.91}$$

(this is the Doppler effect) and an angle relative to the z axis given by

$$\cos\theta' = \frac{\cos\theta - \beta}{1-\beta\cos\theta} \tag{3.92}$$

(this is aberration). Find θ' explicitly for $|\beta| \ll 1$.

22. By writing the angle relation (3.92) as

$$\cos\theta - \cos\theta' = \beta(1-\cos\theta\cos\theta')\ , \tag{3.93}$$

show that

$$\tan\frac{1}{2}\theta' = \sqrt{\frac{1+\beta}{1-\beta}}\tan\frac{1}{2}\theta\ , \tag{3.94}$$

or, replacing the angle θ for the direction of travel by the angle $\alpha = \pi - \theta$ for the direction of arrival,

$$\tan\frac{1}{2}\alpha' = \sqrt{\frac{1-\beta}{1+\beta}}\tan\frac{1}{2}\alpha\ . \tag{3.95}$$

23. An ellipse of eccentricity β is inscribed in a circle. The major axis of the ellipse lies along the x axis, the origin of which is the center of the circle. A line drawn from the origin to a point on the circle makes an angle α with the x axis. Now one finds a related point on the ellipse by moving down, perpendicularly to the x axis, from the point on the circle. A line drawn from the left-hand focus of the ellipse to this point on the ellipse makes an angle α' with the x axis. Show that the relation between α and α' is that of (3.95).
24. Show that the 4-potential produced by a charged particle with 4-velocity v^μ is

$$\begin{aligned} A^\mu(x) &= \frac{1}{4\pi}\int \mathrm{d}s'\eta(x^0 - x^{0\prime}(s'))2\delta[(x-x'(s'))^2]ev^\mu(s') \\ &= -\frac{e}{4\pi}\frac{v^\mu(s')}{(x-x'(s'))v(s')}\ . \end{aligned} \tag{3.96}$$

Here η is the unit step function (the corresponding capital letter looks like the initial letter of Heaviside)

$$\eta(x) = \begin{cases} 1\ , & x > 0\ , \\ 0\ , & x < 0\ . \end{cases} \tag{3.97}$$

Make explicit what is left implicit in the result (3.96). Write the result in $3+1$ dimensional notation and compare with the Liénard-Wiechert potentials (1.125) and (1.126).

25. A charge at rest scatters radiation with unchanged frequency. Give a relativistically invariant form to this statement. Then deduce that radiation of frequency ω_0, moving in the direction of unit vector $\mathbf{n}_0$, which is scattered by a charge with velocity $\mathbf{v}$ into the direction of unit vector $\mathbf{n}$, has the frequency
$$\omega = \omega_0 \frac{1 - \mathbf{n}_0 \cdot \frac{\mathbf{v}}{c}}{1 - \mathbf{n} \cdot \frac{\mathbf{v}}{c}} \ . \tag{3.98}$$
26. Find the total scattering cross section for the scattering of radiation by a charge that is moving with velocity $\mathbf{v}$ in the direction of the incident radiation. Assume that $\beta = v/c$ is small so that is an essentially nonrelativistic calculation.
27. Repeat the above using a relativistic calculational method. Check the consistency of the result with that of the $\beta \ll 1$ calculation.
28. The classical statement that light is scattered with unchanged frequency by a charge at rest appears generally as, in four-vector notation,
$$v^2 = -c^2 \ : \qquad kv = k_0 v \ , \qquad \text{or} \quad v(k - k_0) = 0 \ . \tag{3.99}$$
Now think of a *photon* scattered by a charged particle. If initial momenta are denoted by a subscript 0, the statement of energy-momentum conservation reads
$$(\hbar k + p)^\mu = (\hbar k_0 + p_0)^\mu \ , \tag{3.100}$$
where
$$k^2 = k_0^2 = 0 \ , \qquad p^2 = p_0^2 = -m_0^2 \ , \tag{3.101}$$
where m_0 is the rest mass of the particle. Show that
$$(p + p_0)(k - k_0) = 0 \ , \tag{3.102}$$
or in terms of 4-velocities, given by $p^\mu = m_0 v^\mu$, $p_0^\mu = m_0 v_0^\mu$, that
$$\frac{1}{2}(v + v_0)(k - k_0) = 0 \ . \tag{3.103}$$
Thus the classical result (3.99) appears when the difference between v^μ and v_0^μ can be neglected in
$$\left(\frac{1}{2}(v + v_0)\right)^2 + \left(\frac{1}{2}(v - v_0)\right)^2 = -c^2 \ . \tag{3.104}$$
The nearest quantum equivalent to the classical rest frame occurs when $\mathbf{p}_0 = -\mathbf{p}$. In that frame let $\mathbf{k}_0$ and $\mathbf{k}$ each make an angle $\frac{1}{2}\theta$ with respect to the plane perpendicular to $\mathbf{p}$. Check that (3.102) implies $\omega = \omega_0$. what is the value of $|\mathbf{p}| = |\mathbf{p}_0|$ in terms of ω_0 and the photon scattering angle θ? Under what circumstances can the equal and opposite velocities $\mathbf{v}$ and $\mathbf{v}_0$ be regarded as negligible? (This is the underlying principle of the Free Electron Laser [13].)

29. Integrate the invariant $(\mathrm{d}k)2\delta(k^2)$, where $(\mathrm{d}k) = \mathrm{d}k^0(\mathrm{d}\mathbf{k})$, over all $k^0 > 0$ to arrive at the invariant

$$\frac{(\mathrm{d}\mathbf{k})}{|\mathbf{k}|} = \frac{\omega\,\mathrm{d}\omega}{c^2}\mathrm{d}\Omega\ , \quad \omega = kc\ , \tag{3.105}$$

in which $\mathrm{d}\Omega$ is an element of solid angle. Use the Doppler effect formula (3.91) to deduce the solid angle transformation law,

$$\mathrm{d}\Omega' = \frac{1-\beta^2}{(1-\beta\cos\theta)^2}\mathrm{d}\Omega\ . \tag{3.106}$$

Then get it directly from the aberration formula (3.92). What did you assume about the azimuthal angle ϕ, and why? Check that the above relation is consistent with the requirement that $\int \mathrm{d}\Omega' = 4\pi$.

30. Let v^μ be the four-vector velocity $\gamma(c, \mathbf{v})$, $\gamma = (1-v^2/c^2)^{-1/2}$ of a physical system. Use the invariance of $k^\mu v_\mu$, in relating ω', a frequency observed when the system is at rest, to quantities measured when the system is in motion along the z-axis with velocity v. Compare with a result found in Problem 21. Show that the invariant

$$I = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} - \left(mc\frac{k_\mu \mathrm{d}p^\mu/\mathrm{d}\tau}{k_\nu p^\nu}\right)^2 , \tag{3.107a}$$

is written as

$$I = \left(\frac{E}{mc^2}\right)^2\left[\dot{\mathbf{p}}^2 - \left(\frac{1}{c}\dot{E}\right)^2 - \left(\frac{mc^2}{E}\right)^2\frac{(\mathbf{n}\cdot\dot{\mathbf{p}} - \dot{E}/c)^2}{(1-\mathbf{n}\cdot\mathbf{p}c/E)^2}\right] , \tag{3.107b}$$

which appears in the angular distribution given in (3.111).

31. Verify that the energy radiated per unit time into a unit solid angle, by a system that is momentarily at rest, is given, in any coordinate frame, by the invariant expression

$$-\frac{\mathrm{d}^2p^\mu}{\mathrm{d}\tau\,\mathrm{d}\Omega'}v_\mu\ , \tag{3.108}$$

where v^μ is the velocity 4-vector of the system; $\mathrm{d}\Omega'$ refers to the rest frame. Then use the relation between the momentum and the energy of the radiation moving in a given direction (unit vector $\mathbf{n}$) to write the above radiation quantity, for a system moving with velocity $\mathbf{v}$, as

$$\frac{\mathrm{d}^2E}{\mathrm{d}t\,\mathrm{d}\Omega'}\frac{1-\mathbf{n}\cdot\mathbf{v}/c}{1-v^2/c^2}\ . \tag{3.109}$$

32. The power radiated in the direction $\mathbf{n}$, per unit solid angle $\mathrm{d}\Omega'$, by an accelerated charge that is momentarily at rest, is given by (1.167) or

$$\frac{e^2}{(4\pi)^2c^3}(\mathbf{n}\times\dot{\mathbf{v}})^2 = \frac{e^2}{(4\pi)^2c^3}[\dot{\mathbf{v}}^2 - (\mathbf{n}\cdot\dot{\mathbf{v}})^2]\ . \tag{3.110}$$

Now combine this with the results of (3.109), (3.106), and (3.107b) to produce the power radiated into a solid angle $\mathrm{d}\Omega$,

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{e^2}{(4\pi)^2 m^2 c^3} \left(\frac{mc^2}{E}\right)^2 \left[\frac{\dot{\mathbf{p}}^2 - (\dot{E}/c)^2}{(1 - \mathbf{n}\cdot\mathbf{p}c/E)^3} - \left(\frac{mc^2}{E}\right)^2 \frac{(\mathbf{n}\cdot\dot{\mathbf{p}} - \dot{E}/c)^2}{(1 - \mathbf{n}\cdot\mathbf{p}c/E)^5}\right] . \tag{3.111}$$

This result is the same as that given in (1.210) upon substituting there $\mathbf{v} = \mathbf{p}c^2/E$.