Physics 5573. Electrodynamics I.
Final Examination
Fall 2007

December 14, 2007

Instructions: This examination consists of four problems. If you get stuck on one part, assume a result and proceed onward. Show all your work. Do not hesitate to ask questions. GOOD LUCK!

HAVE A GREAT HOLIDAY BREAK!

1. The action describing a particle of rest mass m_0 interacting with a given vector potential $A^\mu = (\phi, \mathbf{A})$ is

$$W_{12} = \int_2^1 \left[-m_0 c^2 \, d\tau + \frac{e}{c} A_\mu dx^\mu \right],$$

where the 4-velocity of the particle is

$$v^\mu = \frac{dx^\mu}{d\tau},$$

and the proper time element is related to the trajectory of the particle by

$$c \, d\tau = \sqrt{-dx^\mu dx_\mu}.$$

(a) By varying the above action with respect to the particle trajectory,

$$x^\mu \to x^\mu + \delta x^\mu,$$

obtain the equation of motion in terms of a second-order differential equation relating $d^2 x^\mu / d\tau^2$ to the vector potential A^μ and the particle velocity v^μ.

(b) Verify that this equation agrees with Newton's law, with the Lorentz force, in the nonrelativistic limit.

2. Consider a long straight wire, of negligible cross section, carrying a current I. Calculate the magnetic field produced by this wire as follows.

   (a) In the radiation gauge, derive the differential equation satisfied by the vector potential.

   (b) In cylindrical coordinates, with the z axis coinciding with the wire, show that the cylindrically symmetric solution to this equation outside the wire has the form

   $$\mathbf{A} = \hat{\mathbf{z}}(a + b \ln \rho),$$

   where a and b are constants and ρ is the distance from the wire.

   (c) To determine the constant b, integrate the differential equation satisfied by $\mathbf{A}$ found in part 2a over the volume of a circular cylinder concentric with the wire, of radius r and of height h.

   (d) From the $\mathbf{A}$ thus determined, compute the magnetic field $\mathbf{B}$ both in magnitude and direction everywhere outside the wire.

   (e) Verify this result by recasting the magnetostatic equation

   $$\boldsymbol{\nabla} \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}$$

   into an integral equation, Ampére's law, around a closed loop, and solving for $\mathbf{B}$ using symmetry considerations.

3. Consider two current-carrying loops made of wire of negligible thickness, of radius a and b, respectively, parallel to each other, and separated from each other by a distance Z along a line perpendicular to each other, as shown in the figure.

   We are interested in the magnetic energy of interaction between two currents, I_1 and I_2 flowing in loop a and b, respectively. We will only consider the case were the separation distance Z is much greater than the size of either loop, $Z \gg a, b$.

   (a) Show that the magnetic dipole moment of a current loop is $\boldsymbol{\mu} = \frac{I}{c}\mathbf{S}$, where $\mathbf{S}$ is the directed area of the loop.

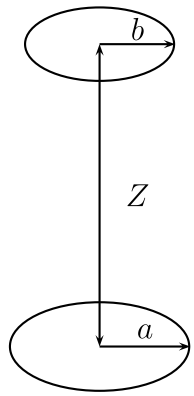

Figure 1: Two parallel circular loops, of radius a and b, are located one above the other and separated by a distance $Z \gg a, b$, along a line perpendicular to the plane of both loops.

(b) Show that the magnetic field produced by a magnetic dipole, far from that dipole, is

$$\mathbf{B} = -\boldsymbol{\nabla}\left(\frac{\boldsymbol{\mu}\cdot\mathbf{r}}{r^3}\right),$$

where $\mathbf{r}$ is the position vector from the dipole to the field point.

(c) Calculate the interaction energy between dipole a and dipole b, the two current loops, from the interaction of a magnetic dipole with an external magnetic field. Give the answer in terms of the given quantities I_1, I_2, a, b, and Z.

(d) What is the force between these two dipoles (current loops)? Is it attractive or repulsive? Give a physical reason for the sign.

4. The Coulomb potential

$$G(\mathbf{r}-\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}, \tag{1}$$

is also referred to as the Coulomb Green's function because it satisfies the equation

$$-\nabla^2 G(\mathbf{r}-\mathbf{r}') = 4\pi\delta(\mathbf{r}-\mathbf{r}'). \tag{2}$$

(a) Given the Fourier representation of a delta function

$$\delta(x - x') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')}, \tag{3}$$

show that the three-dimensional delta function has the representation

$$\delta^3(\mathbf{r} - \mathbf{r}') = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}. \tag{4}$$

(b) Then show from the differential equation (2) satisfied by $G(\mathbf{r} - \mathbf{r}')$, that the Fourier transform of the Green's function, defined by

$$G(\mathbf{r} - \mathbf{r}') = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \tilde{G}(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}, \tag{5}$$

is simply

$$\tilde{G}(\mathbf{k}) = \frac{4\pi}{k^2}. \tag{6}$$

(c) Now insert this result (6) into (5) and carry out the integral by writing

$$\frac{1}{k^2} = \int_0^{\infty} d\lambda \, e^{-\lambda k^2}. \tag{7}$$

(Why is this true?) Carry out the integral over k_x, k_y, and k_z by completing the square and noting that

$$\int_{-\infty}^{\infty} dk \, e^{-\lambda k^2} = \sqrt{\frac{\pi}{\lambda}}. \tag{8}$$

(d) Integrate over λ by changing variables to $\mu = 1/\lambda$, and recognizing that

$$\int_0^{\infty} d\mu \, \mu^{s-1} \, e^{-b\mu} = \Gamma(s) \, b^{-s}, \tag{9}$$

where Γ is the gamma function. In particular, because $\Gamma(1/2) = \sqrt{\pi}$, verify that (1) holds.