Physics 5573. Electrodynamics I.
Second Examination
Fall 2007

November 9, 2007

Instructions: This examination consists of three problems. If you get stuck on one part, assume a result and proceed onward. Show all your work. Do not hesitate to ask questions. GOOD LUCK!

1. A nonrelativistic particle of mass m and charge e moving in a constant magnetic field $\mathbf{B}$ is described by the Hamiltonian:

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2, \quad \mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r}.$$

   (a) Verify that $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$.

   (b) From Hamilton's equations deduce the particle's velocity $\mathbf{v}$.

   (c) From Hamilton's equations deduce the equation of motion for $\mathbf{p}$. Give $d\mathbf{p}/dt$ in terms of $\mathbf{B}$.

   (d) Show that the latter is consistent with the Lorentz force law.

2. Consider an electromagnetic field described by the vector potential ($\mathbf{a}$ is a constant vector)

$$\mathbf{A}(\mathbf{r}) = \frac{\mathbf{a}\times\mathbf{r}}{r^3}, \quad r = |\mathbf{r}|,$$

   while the scalar potential is zero.

   (a) Calculate explicitly the magnetic field $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ for all $r > 0$.

(b) Show that $\mathbf{B}$ can be written in the form

$$\mathbf{B} = \boldsymbol{\nabla}\phi + \mathbf{a}\nabla^2\psi,$$

and identify the functions ϕ and ψ. What is $\nabla^2\psi$?

(c) From the equation of magnetostatics

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J},$$

determine the current density $\mathbf{J}$ belonging to this magnetic field.

(d) Use this current to work out the magnetic dipole moment from

$$\boldsymbol{\mu} = \frac{1}{2c}\int (d\mathbf{r})\mathbf{r} \times \mathbf{J}.$$

[Hint: Integrate by parts.]

3. A charged particle of charge e and mass m subject to a harmonic restoring force and a damping force, under the influence of an external electric field $\mathbf{E}(t)$, is described by the equation

$$\left(\frac{d^2}{dt^2} + \omega_0^2 + \gamma\frac{d}{dt}\right)\mathbf{r}(t) = \frac{e}{m}\mathbf{E}(t).$$

If we multiply this equation by ne, where n is the number density of such particles, we obtain a differential equation for the electric polarization $\mathbf{P}$.

(a) Find a solution of the above-referred-to equation in the form

$$\mathbf{P}(t) = \int_{-\infty}^{\infty} dt'\chi(t-t')\mathbf{E}(t'),$$

and write down the differential equation satisfied by $\chi(t-t')$. [Hint: the inhomogeneous term in this equation is proportional to a δ-function.]

(b) Solve this equation by requiring that $\chi(t-t') = 0$ if $t - t' < 0$ (causality). Show that for $t - t' > 0$ the solution has the form

$$\chi(t-t') = ae^{i\lambda_+(t-t')} + be^{i\lambda_-(t-t')},$$

where a and b are constants. What are the (constant) complex numbers $\lambda_\pm$?

(c) Determine the constants a and b by requiring χ be continuous at $t - t' = 0$, but χ has a discontinuity in its first derivative at $t - t' = 0$ dictated by the equation of motion it satisfies. [Hint: Integrate the differential equation in time found in part 3a over a small interval in t including the point t'.] Solve the resulting system of linear equations for a and b, and give an explicit form for $\chi(t - t')$.

(d) Alternatively, recall that in class we solved this problem by a Fourier transform, leading to a dielectric constant

$$\varepsilon(\omega) = 1 + 4\pi\chi(\omega).$$

Explicitly, we found

$$\varepsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}, \quad \omega_p^2 = \frac{4\pi n e^2}{m}.$$

To connect this to the above, we simply have to carry out a Fourier transform,

$$\chi(t - t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \chi(\omega) e^{-i\omega(t-t')}.$$

Show that indeed, $\chi(t - t') = 0$ for $t < t'$, but that for $t > t'$ you recover the same result found in part 3c. [Hint: Where are the poles in $\chi(\omega)$?]