# Physics 5573. Electrodynamics I.
# Second Examination
# Fall 2003

November 21, 2003

**Instructions:** This examination consists of two problems. If you get stuck on one part, assume a result and proceed onward. Show all your work. Do not hesitate to ask questions. GOOD LUCK!

1. The Lagrangian of a relativistic particle of mass $m$ and charge $e$ interacting with the electromagnetic field is

$$L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} - \int (d\mathbf{r}) \left( \rho\phi - \frac{1}{c}\mathbf{j} \cdot \mathbf{A} \right) + \int (d\mathbf{r}) \frac{E^2 - B^2}{8\pi},$$

where

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \quad \mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A} - \boldsymbol{\nabla}\phi, \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}.$$

The action

$$W_{01} = \int_{t_0}^{t_1} dt\, L$$

is stationary with respect to small variations in the dynamical variables about the true trajectory, when the variations vanish at the endpoints.

(a) By varying $\mathbf{r}$, $\mathbf{r} \to \mathbf{r} + \delta\mathbf{r}$, derive the equation of motion for $\mathbf{r}$.

(b) Show that this result is equivalent to the Lorentz force law in terms of the rate of change of the relativistic momentum $m_0\gamma\mathbf{v}$.

(c) By varying $\phi$, obtain Gauss' law.

(d) By varying $\mathbf{A}$, obtain Ampere's law, including Maxwell's displacement current.

(e) Obtain the Hamiltonian for this system by making a change in the time integration parameter, $t \to t + \delta t$, and identifying $H$ from

$$\delta W = -\int_{t_0}^{t_1} dt\, \frac{d\delta t}{dt} H.$$

(Note that variations resulting from the dependence of the dynamical variables, $\mathbf{r}$, $\phi$, and $\mathbf{A}$ on $t$ vanish by virtue of the stationary action principle.) *Alternatively*, obtain the Hamiltonian from

$$H = \int (d\mathbf{r}) \frac{\delta L}{\delta \dot{\mathbf{A}}} \cdot \dot{\mathbf{A}} + \frac{\partial L}{\partial \mathbf{v}} \cdot \mathbf{v} - L.$$

2. Consider a plane electromagnetic wave impinging normally from vacuum onto a flat dielectric (permittivity $\epsilon$) surface. Let the direction of propagation of this wave be the $+z$ direction, and the interface lie in the $x$-$y$ plane at $z = 0$.

(a) If the incident plane wave has the form ($z < 0$)

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0\, e^{ikz - i\omega t},$$
$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0\, e^{ikz - i\omega t},$$

give the relation between the amplitudes of $\mathbf{E}_0$ and $\mathbf{B}_0$, and between their directions.

(b) At the interface, some of the wave is reflected and some is transmitted. Because of the normal incidence of the incoming wave, the transmitted wave propagates along the $+z$ direction, while the reflected wave propagates in the $-z$ direction. If the reflected wave has magnetic amplitude $r\mathbf{B}_0$, where $r$ is the reflection coefficient to be determined below, what is the amplitude of the transmitted magnetic field? (Hint: the tangential magnetic field is continuous at the interface between two dielectric bodies.)

(c) What are the amplitudes of the transmitted and reflected electric fields?

(d) By adding the electric and the magnetic fields on the two sides of the interface, calculate the energy flux vector on both sides in terms of the initial magnetic amplitude $\mathbf{B}_0$. (Assume $r$ is real.)

(e) Use energy conservation at the interface to obtain an equation for $r$, the reflection amplitude, in terms of $\epsilon$. Solve this equation for $r$. Is this result physically reasonable? What happens as $\epsilon \to 1$, $\epsilon \to \infty$? (The latter limit should correspond to the $z > 0$ region being a perfect conductor.)