4

Waveguides and Equivalent Transmission Lines

A waveguide is a device for transferring electromagnetic energy from one point to another without appreciable loss. In its simplest form it consists of a hollow metallic tube of rectangular or circular cross section, within which electromagnetic waves can propagate. The two conductor transmission line discussed in the Appendix is a particular type of waveguide, with special properties. The simple physical concept implied by these examples may be extended to include any region within which one-dimensional propagation of electromagnetic waves can occur. It is the purpose of this chapter to establish the theory of various types of simple waveguides, expressed in the general transmission-line nomenclature developed in the Appendix.

This chapter will be devoted to the theory of uniform waveguides – metallic tubes which have the same cross-section in any plane perpendicular to the axis of the guide. Initially, the simplifying assumption will be made that the metallic walls of the waveguide are perfectly conducting. Since the field is then entirely confined to the interior of the waveguide, the guide is completely described by specifying the curve C which defines a cross-section σ of the inner waveguide surface S. The curve C may be a simple closed curve, corresponding to a hollow waveguide, or two unconnected curves, as in a coaxial line.

4.1 Transmission-Line Formulation

We first consider the problem of finding the possible fields that can exist within a waveguide, in the absence of any impressed currents. This is equivalent to seeking the solutions of the Maxwell equations

$$\boldsymbol{\nabla} \times \mathbf{E} = \mathrm{i}k\zeta\mathbf{H} \,, \tag{4.1a}$$

$$\boldsymbol{\nabla} \times \mathbf{H} = -\mathrm{i}k\eta\mathbf{E} \,, \tag{4.1b}$$

where we have defined [SI units]

$$k = \omega\sqrt{\varepsilon\mu} = \frac{\omega}{c} \ , \tag{4.2a}$$

c being the speed of light in the medium inside the guide, and introduced the abbreviations

$$\zeta = \sqrt{\frac{\mu}{\varepsilon}} \ , \quad \eta = \sqrt{\frac{\varepsilon}{\mu}} = \zeta^{-1} \ . \tag{4.2b}$$

These equations are to be solved subject to the boundary condition

$$\mathbf{n} \times \mathbf{E} = \mathbf{0} \quad \text{on } S \ , \tag{4.3}$$

where $\hat{\mathbf{n}}$ is the unit normal to the surface S of the guide. Recall that the other two Maxwell equations, in charge-free regions,

$$\boldsymbol{\nabla} \cdot \mathbf{D} = 0 \ , \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \ , \tag{4.4}$$

are contained within these equations, as is the boundary condition $\mathbf{n} \cdot \mathbf{B} = 0$. The medium filling the waveguide is assumed to be uniform and non-dissipative. In view of the cylindrical nature of the boundary surface, it is convenient to separate the field equations into components parallel to the axis of the guide, which we take as the z axis, and components transverse to the guide axis. This we achieve by scalar and vector multiplication with $\mathbf{e}$, a unit vector in the z direction, thus obtaining

$$\boldsymbol{\nabla} \cdot \mathbf{e} \times \mathbf{E} = -\mathrm{i}k\zeta H_z \ , \tag{4.5a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{e} \times \mathbf{H} = \mathrm{i}k\eta E_z \ , \tag{4.5b}$$

and

$$\boldsymbol{\nabla} E_z - \frac{\partial}{\partial z}\mathbf{E} = \mathrm{i}k\zeta \mathbf{e} \times \mathbf{H} \ , \tag{4.6a}$$

$$\boldsymbol{\nabla} H_z - \frac{\partial}{\partial z}\mathbf{H} = -\mathrm{i}k\eta \mathbf{e} \times \mathbf{E} \ . \tag{4.6b}$$

On substituting (4.5b) [(4.5a)] into (4.6a) [(4.6b)], one recasts the latter into the form

$$\frac{\partial}{\partial z}\mathbf{E} = \mathrm{i}k\zeta \left(\mathbf{1} + \frac{1}{k^2}\boldsymbol{\nabla}\boldsymbol{\nabla}\right) \cdot \mathbf{H} \times \mathbf{e} \ , \tag{4.7a}$$

$$\frac{\partial}{\partial z}\mathbf{H} = \mathrm{i}k\eta \left(\mathbf{1} + \frac{1}{k^2}\boldsymbol{\nabla}\boldsymbol{\nabla}\right) \cdot \mathbf{e} \times \mathbf{E} \ , \tag{4.7b}$$

in which $\mathbf{1}$ denotes the unit dyadic. This set of equations is fully equivalent to the original field equations, for it still contains (4.5a), (4.5b) as its z component. The transverse components of (4.7a), (4.7b) constitute a system of differential equations to determine the transverse components of the electric and magnetic fields. These equations are in transmission line form, but with the series impedance and the shunt admittance per unit length appearing as dyadic differential operators. The subsequent analysis has for its aim the replacement of the operator transmission line equations by an infinite set of ordinary differential equations. This is performed by successively suppressing the vectorial aspect of the equations and the explicit dependence on x and y, the coordinates in a transverse plane.

Any two-component vector field, such as the transverse part of the electric field $\mathbf{E}_\perp$, can be represented as a linear combination of two vectors derived from a potential function and a stream function, respectively. Thus

$$\mathbf{E}_\perp = -\boldsymbol{\nabla}_\perp V' + \mathbf{e} \times \boldsymbol{\nabla} V'' , \tag{4.8}$$

where $V'(\mathbf{r})$ and $V''(\mathbf{r})$ are two arbitrary scalar functions and $\boldsymbol{\nabla}_\perp$ indicates the transverse part of the gradient operator. In a similar way, we write

$$\mathbf{H}_\perp = -\mathbf{e} \times \boldsymbol{\nabla} I' - \boldsymbol{\nabla}_\perp I'' , \tag{4.9a}$$

or

$$\mathbf{H} \times \mathbf{e} = -\boldsymbol{\nabla}_\perp I' + \mathbf{e} \times \boldsymbol{\nabla} I'' , \tag{4.9b}$$

with $I'(\mathbf{r})$ and $I''(\mathbf{r})$ two new arbitrary scalar functions. This general representation can be obtained by constructing the two-component characteristic vectors (eigenvectors) of the operator $\mathbf{1} + \frac{1}{k^2}\boldsymbol{\nabla\nabla}$. Such vectors must satisfy the eigenvector equation in the form

$$\boldsymbol{\nabla}_\perp \boldsymbol{\nabla} \cdot \mathbf{A}_\perp = \gamma \mathbf{A}_\perp . \tag{4.10}$$

Hence, either $\boldsymbol{\nabla} \cdot \mathbf{A}_\perp = 0$ and $\gamma = 0$, implying that $\mathbf{A}_\perp$ is the curl of a vector directed along the z axis; or $\boldsymbol{\nabla} \cdot \mathbf{A}_\perp \neq 0$, and $\mathbf{A}_\perp$ is the gradient of a scalar function. The most general two-component vector $\mathbf{A}_\perp$ is a linear combination of these two types, and $\mathbf{e} \times \mathbf{A}$ is still of the same form, as it must be. In consequence of these observations, the substitution of the representation (4.8), (4.9a) into the differential equations (4.7a), (4.7b) will produce a set of equations in which every term has one or the other of these forms. This yields a system of four scalar differential equations, which are grouped into two pairs,

$$\frac{\partial}{\partial z} I' = \mathrm{i}k\eta V' , \quad \frac{\partial}{\partial z} V' = \mathrm{i}k\zeta \left(1 + \frac{1}{k^2}\nabla_\perp^2\right) I' , \tag{4.11a}$$

$$\frac{\partial}{\partial z} I'' = \mathrm{i}k\eta \left(1 + \frac{1}{k^2}\nabla_\perp^2\right) V'' , \quad \frac{\partial}{\partial z} V'' = \mathrm{i}k\zeta I'' , \tag{4.11b}$$

where $\nabla_\perp^2$ is the Laplacian for the transverse coordinates x and y. (Any constant annihilated by $\boldsymbol{\nabla}_\perp$ is excluded because it would not contribute to the electric and magnetic fields.) The longitudinal field components may now be written as

$$\mathrm{i}k\eta E_z = \nabla_\perp^2 I' , \tag{4.12a}$$

$$\mathrm{i}k\zeta H_z = \nabla_\perp^2 V'' . \tag{4.12b}$$

The net effect of these operations is the decomposition of the field into two independent parts derived, respectively, from the scalar functions V', I', and V'', I''. Note that the first type of field in general possesses a longitudinal component of electric field, but no longitudinal magnetic field, while the situation is reversed with the second type of field. For this reason, the various field configurations derived from V' and I' are designated as E modes, while those obtained from V'' and I'' are called H modes; the nomenclature in each case specifies the non-vanishing z component of the field.[1]

The scalar quantities involved in (4.11a), (4.11b) are functions of x, y, and z. The final step in the reduction to one-dimensional equations consists in representing the x, y dependence of these functions by an expansion in the complete set of functions forming the eigenfunctions of $\nabla_\perp^2$. For the E mode, let these functions be $\varphi_a(x, y)$, satisfying

$$(\nabla_\perp^2 + \gamma_a'^2)\varphi_a(x, y) = 0 \; , \tag{4.13}$$

and subject to boundary conditions, which we shall shortly determine. On substituting the expansion

$$V'(x, y, z) = \sum_a \varphi_a(x, y) V_a'(z) \; , \tag{4.14a}$$

$$I'(x, y, z) = \sum_a \varphi_a(x, y) I_a'(z) \; , \tag{4.14b}$$

into (4.11a), we immediately obtain the transmission line equations

$$\frac{\mathrm{d}}{\mathrm{d}z} I_a'(z) = \mathrm{i}k\eta V_a'(z) \; , \tag{4.15a}$$

$$\frac{\mathrm{d}}{\mathrm{d}z} V_a'(z) = \mathrm{i}k\zeta \left(1 - \frac{\gamma_a'^2}{k^2}\right) I_a'(z) \; . \tag{4.15b}$$

In a similar way, we introduce another set of eigenfunctions for $\nabla_\perp^2$:

$$(\nabla_\perp^2 + \gamma_a''^2)\psi_a(x, y) = 0 \; . \tag{4.16}$$

and expand the H mode quantities in terms of them:

$$V''(x, y, z) = \sum_a \psi_a(x, y) V_a''(z) \; , \tag{4.17a}$$

$$I''(x, y, z) = \sum_a \psi_a(x, y) I_a''(z) \; . \tag{4.17b}$$

The corresponding differential equations are

[1] A more common terminology for E modes are TM modes, meaning "transverse magnetic"; and for H modes, TE modes, for "transverse electric." Still another notation is $\perp$ for "perpendicular," referring to H modes, and $\|$ for "parallel," referring to E modes.

$$\frac{\mathrm{d}}{\mathrm{d}z} I_a''(z) = \mathrm{i}k\eta \left(1 - \frac{\gamma_a''^2}{k^2}\right) V_a''(z) \;, \tag{4.18a}$$

$$\frac{\mathrm{d}}{\mathrm{d}z} V_a''(z) = \mathrm{i}k\zeta I_a''(z) \;. \tag{4.18b}$$

The boundary conditions on the electric field require that

$$E_z = 0, \quad E_s = 0 \quad \text{on} \quad S \;, \tag{4.19}$$

where E_s is the component of the electric field tangential to the boundary curve C. These conditions imply that

$$\nabla_\perp^2 I' = 0, \quad \frac{\partial}{\partial s} V' = \frac{\partial}{\partial n} V'' = 0 \quad \text{on} \quad S \;, \tag{4.20}$$

where $\frac{\partial}{\partial n}$ is the derivative normal to the surface of the waveguide S, and $\frac{\partial}{\partial s}$ is the circumferential derivative, tangential to the curve C. Since these equations must be satisfied for all z, they impose the following requirements on the functions φ_a and ψ_a:

$$\gamma_a'^2 \varphi_a = 0 \;, \quad \frac{\partial}{\partial s} \varphi_a = 0 \;, \quad \frac{\partial}{\partial n} \psi_a = 0 \quad \text{on} \quad C \;. \tag{4.21}$$

If we temporarily exclude the possibility $\gamma_a' = 0$, the second E mode boundary condition is automatically included in the first statement, that $\varphi_a = 0$ on the boundary curve C. Hence, E modes are derived from scalar functions defined by

$$\left(\nabla_\perp^2 + \gamma_a'^2\right) \varphi_a(x, y) = 0 \;, \tag{4.22a}$$

$$\varphi_a(x, y) = 0 \quad \text{on} \quad C \;, \tag{4.22b}$$

while H modes are derived from functions satisfying

$$\left(\nabla_\perp^2 + \gamma_a''^2\right) \psi_a(x, y) = 0 \;, \tag{4.23a}$$

$$\frac{\partial}{\partial n} \psi_a(x, y) = 0 \quad \text{on} \quad C \;, \tag{4.23b}$$

These equations are often encountered in physics. For example, they describe the vibrations of a membrane bounded by the curve C, which is either rigidly clamped at the boundary [(4.22b)], or completely free [(4.23b)]. Mathematically, these are referred to a Dirichlet and Neumann boundary conditions, respectively. Each equation defines an infinite set of eigenfunctions and eigenvalues φ_a, γ_a' and ψ_a, γ_a''. Hence, a waveguide possesses a two-fold infinity of possible modes of electromagnetic oscillation, each completely characterized by one of these scalar functions and its attendant eigenvalue.

We shall now show that the discarded possibility, $\gamma_a' = 0$, cannot occur for hollow waveguides, but does correspond to an actual field configuration in two conductor lines, being in fact the T mode discussed in the Appendix. The scalar function φ associated with $\gamma'_a = 0$ satisfies Laplace's equation

$$\nabla_\perp^2 \varphi(x,y) = 0 \ , \tag{4.24}$$

and is restricted by the second boundary condition, $\frac{\partial}{\partial s}\varphi(x,y) = 0$ on C, or

$$\varphi(x,y) = \text{ constant on } C \ . \tag{4.25}$$

Since φ satisfies Laplace's equation, we deduce that

$$\int_C \mathrm{d}s \, \varphi \frac{\partial}{\partial n}\varphi = \int_\sigma \mathrm{d}\sigma \, (\boldsymbol{\nabla}_\perp \varphi)^2 \ , \tag{4.26}$$

in which the line integral is taken around the curve C and the surface integral is extended over the guide cross-section σ. For a hollow waveguide with a cross-section bounded by a single closed curve on which

$$\varphi = \text{constant} = \varphi_0 \ , \tag{4.27}$$

we conclude

$$\int_C \mathrm{d}s \, \varphi \frac{\partial}{\partial n}\varphi = \varphi_0 \int_C \mathrm{d}s \, \frac{\partial}{\partial n}\varphi = \varphi_0 \int_\sigma \mathrm{d}\sigma \, \nabla_\perp^2 \varphi = 0 \ , \tag{4.28}$$

and therefore from (4.26) $\boldsymbol{\nabla}_\perp \varphi = 0$ everywhere within the guide, which implies that all field components vanish, effectively denying the existence of such a mode. If, however, the contour C consists of two unconnected curves C_1 and C_2, as in a coaxial line, the boundary condition, $\frac{\partial}{\partial s}\varphi = 0$ on C, requires that φ be constant on each contour

$$\varphi = \varphi_1 \text{ on } C_1, \quad \varphi = \varphi_2 \text{ on } C_2 \ , \tag{4.29}$$

but does not demand that $\varphi_1 = \varphi_2$. Hence

$$\int_C \mathrm{d}s \, \varphi \frac{\partial}{\partial n}\varphi = \varphi_1 \int_{C_1} \mathrm{d}s \frac{\partial}{\partial n}\varphi + \varphi_2 \int_{C_2} \mathrm{d}s \frac{\partial}{\partial n}\varphi = (\varphi_1 - \varphi_2) \int_{C_1} \mathrm{d}s \frac{\partial}{\partial n}\varphi \ , \tag{4.30}$$

since

$$0 = \int_C \mathrm{d}s \frac{\partial}{\partial n}\varphi = \int_{C_1} \mathrm{d}s \frac{\partial}{\partial n}\varphi + \int_{C_2} \mathrm{d}s \frac{\partial}{\partial n}\varphi \ , \tag{4.31}$$

and the preceding proof fails if $\varphi_1 \neq \varphi_2$. The identification with the T mode is completed by noting [(4.12a)] that $E_z = H_z = 0$.

The preceding discussion has shown that the electromagnetic field within a waveguide consists of a linear superposition of an infinite number of completely independent field configurations, or modes. Each mode has a characteristic field pattern across any section of the guide, and the amplitude variations of the fields along the guide are specified by "currents" and "voltages" which satisfy transmission-line equations. We shall summarize our results by collecting together the fundamental equations describing a typical E mode and H mode (omitting distinguishing indices for simplicity).

- E mode:

$$\mathbf{E}_\perp = -\boldsymbol{\nabla}_\perp \varphi(x,y)V(z)\;, \tag{4.32a}$$
$$\mathbf{H}_\perp = -\mathbf{e}\times\boldsymbol{\nabla}\varphi(x,y)I(z)\;, \tag{4.32b}$$
$$E_z = \mathrm{i}\zeta\frac{\gamma^2}{k}\varphi(x,y)I(z)\;, \tag{4.32c}$$
$$H_z = 0\;, \tag{4.32d}$$
$$(\nabla_\perp^2+\gamma^2)\varphi(x,y) = 0, \quad \varphi(x,y) = 0 \text{ on } C\;, \tag{4.32e}$$
$$\frac{\mathrm{d}}{\mathrm{d}z}I(z) = \mathrm{i}k\eta V(z)\;, \tag{4.32f}$$
$$\frac{\mathrm{d}}{\mathrm{d}z}V(z) = \mathrm{i}k\zeta\left(1-\frac{\gamma^2}{k^2}\right)I(z)\;. \tag{4.32g}$$

- H mode:

$$\mathbf{E}_\perp = \mathbf{e}\times\boldsymbol{\nabla}\psi(x,y)V(z)\;, \tag{4.33a}$$
$$\mathbf{H}_\perp = -\boldsymbol{\nabla}_\perp\psi(x,y)I(z)\;, \tag{4.33b}$$
$$E_z = 0\;, \tag{4.33c}$$
$$H_z = \mathrm{i}\eta\frac{\gamma^2}{k}\psi(x,y)V(z)\;, \tag{4.33d}$$
$$(\nabla_\perp^2+\gamma^2)\psi(x,y) = 0, \quad \frac{\partial}{\partial n}\psi(x,y) = 0 \text{ on } C\;, \tag{4.33e}$$
$$\frac{\mathrm{d}}{\mathrm{d}z}I(z) = \mathrm{i}k\eta\left(1-\frac{\gamma^2}{k^2}\right)V(z)\;, \tag{4.33f}$$
$$\frac{\mathrm{d}}{\mathrm{d}z}V(z) = \mathrm{i}k\zeta I(z)\;. \tag{4.33g}$$

The T mode in a two-conductor line is to be regarded as an E mode with $\gamma = 0$, and the boundary condition replaced by $\frac{\partial}{\partial s}\varphi = 0$. It may also be considered an H mode with $\gamma = 0$.

The transmission line equations for the two mode types, written as

- E mode:

$$\frac{\mathrm{d}}{\mathrm{d}z}I(z) = \mathrm{i}\omega\varepsilon V(z)\;, \tag{4.34a}$$
$$\frac{\mathrm{d}}{\mathrm{d}z}V(z) = \left(\mathrm{i}\omega\mu+\frac{\gamma^2}{\mathrm{i}\omega\varepsilon}\right)I(z)\;, \tag{4.34b}$$

- H mode:

$$\frac{\mathrm{d}}{\mathrm{d}z}I(z) = \left(\mathrm{i}\omega\varepsilon+\frac{\gamma^2}{\mathrm{i}\omega\mu}\right)V(z)\;, \tag{4.35a}$$
$$\frac{\mathrm{d}}{\mathrm{d}z}V(z) = \mathrm{i}\omega\mu I(z)\;, \tag{4.35b}$$

are immediately recognized as the equations of the E amd H type for the distributed parameter circuits discussed in the Appendix. The E mode equivalent transmission line has distributed parameters per unit length specified by a shunt capacitance $C = \varepsilon$, and series inductance $L = \mu$, and a series capacitance $C' = \varepsilon/\gamma^2$. The H mode line distributed parameters are a series inductance $L = \mu$, a shunt capacitance $C = \varepsilon$, and a shunt inductance $L'' = \mu/\gamma^2$, all per unit length.[2] Thus, if we consider a plane wave, $I \propto \mathrm{e}^{\mathrm{i}\kappa z}$, with $V = ZI$, the propagation constant κ and characteristic impedance $Z = 1/Y$ associated with the two types of lines are

- E mode:

$$\kappa = \sqrt{k^2 - \gamma^2} \,, \tag{4.36a}$$

$$Z = \zeta \frac{\kappa}{k} \,, \tag{4.36b}$$

- H mode:

$$\kappa = \sqrt{k^2 - \gamma^2} \,, \tag{4.36c}$$

$$Y = \eta \frac{\kappa}{k} \,. \tag{4.36d}$$

We may again remark on the filter property of these transmission lines, which is discussed in the Appendix. Actual transport of energy along a waveguide in a particular mode can only occur if the wave number k exceeds the quantity γ associated with the mode. The eigenvalue γ is therefore referred to as the cutoff or critical wavenumber for the mode. Other quantities related to the cutoff wavenumber are the cutoff wavelength,

$$\lambda_c = \frac{2\pi}{\gamma} \,, \tag{4.37}$$

and the cutoff (angular) frequency

$$\omega_c = \gamma(\epsilon\mu)^{-1/2} \,. \tag{4.38}$$

When the frequency exceeds the cutoff frequency for a particular mode, the wave motion on the transmission line, indicating the field variation along the guide, is described by an associated wavelength

$$\lambda_g = \frac{2\pi}{\kappa} \,, \tag{4.39}$$

which is called the guide wavelength. The relation between the guide wavelength, intrinsic wavelength, and cutoff wavelength for a particular mode is, according to (4.36a),

[2] The vacuum value of the universal series inductance and shunt capacitance is $L_0 = \mu_0 = 1.257\,\mu\mathrm{H/m}$ and $C_0 = \varepsilon_0 = 8.854\,\mathrm{pF/m}$, respectively.

$$\frac{1}{\lambda_g} = \sqrt{\frac{1}{\lambda^2} - \frac{1}{\lambda_c^2}} \,, \tag{4.40a}$$

or

$$\lambda_g = \frac{\lambda}{\sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2}} \,. \tag{4.40b}$$

Thus, at cutoff ($\lambda = \lambda_c$), the guide wavelength is infinite and becomes imaginary at longer wavelengths, indicating attenuation, while at very short wavelengths ($\lambda \ll \lambda_c$), the guide wavelength is substantially equal to the intrinsic wavelength of the guide medium. Correspondingly, the characteristic impedance for a E (H) mode is zero (infinite) at the cutoff frequency and is imaginary at lower frequencies in the manner typical of a capacitance (inductance). The characteristic impedance approaches the intrinsic impedance of the medium $\sqrt{\mu/\varepsilon}$ for very short wavelengths. For $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, the latter reduces to the impedance of free space,

$$Z_0 = \frac{1}{Y_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.6\,\Omega \,. \tag{4.41}$$

The existence of a cutoff frequency for each mode involves the implicit statement that γ^2 is real and positive; γ is positive by definition. A proof is easily supplied for both E and H modes with the aid of the identity

$$\int_C \mathrm{d}s\, f^* \frac{\partial}{\partial n} f = \int_\sigma \mathrm{d}\sigma\, |\boldsymbol{\nabla}_\perp f|^2 - \gamma^2 \int_\sigma \mathrm{d}\sigma\, |f|^2 \,, \tag{4.42}$$

where f stands for either an E-mode function φ or an H-mode function ψ. In either event, the line integral vanishes and

$$\gamma^2 = \frac{\int_\sigma \mathrm{d}\sigma\, |\boldsymbol{\nabla}_\perp f|^2}{\int_\sigma \mathrm{d}\sigma\, |f|^2} \,, \tag{4.43}$$

which establishes the theorem. It may be noted that we have admitted, in all generality, that f may be complex. However, with the knowledge that γ^2 is real, it is evident from the form of the defining wave equation and boundary conditions that real mode functions can always be chosen.

The impedance (admittance) at a given point on the transmission line describing a particular mode,

$$Z(z) = \frac{1}{Y(z)} = \frac{V(z)}{I(z)} \,, \tag{4.44}$$

determines the ratio of the transverse electric and magnetic field components at that point. According to (4.32a) and (4.32b), an E mode magnetic field is related to the transverse electric field by

$$\text{E mode:} \quad \mathbf{H} = Y(z)\mathbf{e} \times \mathbf{E} \,, \tag{4.45a}$$

which is a general vector relation since it correctly predicts that $H_z = 0$. The analogous H-mode relation is

$$\text{H mode:} \quad \mathbf{E} = -Z(z)\mathbf{e} \times \mathbf{H} , \tag{4.45b}$$

For either type of mode, the connections between the rectangular components of the transverse fields are

$$E_x = Z(z)H_y, \quad E_y = -Z(z)H_x , \tag{4.46a}$$

$$H_x = -Y(z)E_y, \quad H_y = Y(z)E_x . \tag{4.46b}$$

In the particular case of a progressive wave propagating (or attenuating) in the positive z direction, the impedance at every point equals the characteristic impedance of the line, $Z(z) = Z$. The analogous relation $Z(z) = -Z$ describes a wave progressing in the negative direction.

4.2 Hertz Vectors

The reduction of the vector field equations to a set of transmission-line equations, as set forth in Sec. 4.1, requires four scalar functions of z for its proper presentation. However, it is often convenient to eliminate two of these functions and exhibit the general electromagnetic field as derived from two scalar functions of position, which appear in the role of single component Hertz vectors. On eliminating the functions $V'(\mathbf{r})$ amd $I''(\mathbf{r})$ with the aid of (4.15a) and (4.18b), the transverse components of $\mathbf{E}$ and $\mathbf{H}$, (4.8), (4.9a), become

$$\mathbf{E}_\perp = \frac{\mathrm{i}}{k}\zeta\boldsymbol{\nabla}_\perp\frac{\partial}{\partial z}I' + \mathbf{e} \times \boldsymbol{\nabla}V'' , \tag{4.47a}$$

$$\mathbf{H}_\perp = -\mathbf{e} \times \boldsymbol{\nabla}I' + \frac{\mathrm{i}}{k}\eta\boldsymbol{\nabla}_\perp\frac{\partial}{\partial z}V'' , \tag{4.47b}$$

which can be combined with the expressions for the longitudinal field components, (4.12a), (4.12b), into general vector equations

$$\mathbf{E} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\Pi}') + \mathrm{i}k\zeta\boldsymbol{\nabla} \times \boldsymbol{\Pi}'' , \tag{4.48a}$$

$$\mathbf{H} = -\mathrm{i}k\eta\boldsymbol{\nabla} \times \boldsymbol{\Pi}' + \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\Pi}'') , \tag{4.48b}$$

The electric and magnetic Hertz vectors that appear in this formulation only posses z components, which are given by

$$\Pi'_z = \frac{\mathrm{i}}{k}\zeta I' , \quad \Pi''_z = \frac{\mathrm{i}}{k}\eta V'' . \tag{4.49}$$

The Maxwell equations are completely satisfied if the Hertz vector components satisfy the scalar wave equation:

$$(\nabla^2 + k^2)\Pi'_z = 0 , \quad (\nabla^2 + k^2)\Pi''_z = 0 , \tag{4.50}$$

which is verified by eliminating V' and I'' from (4.11a), (4.11b). For a particular E mode, the scalar function I' is proportional to the longitudinal electric field and

$$\text{E mode:} \quad \Pi'_z = \frac{1}{\gamma^2} E_z \ . \tag{4.51a}$$

Similarly,

$$\text{H mode:} \quad \Pi''_z = \frac{1}{\gamma^2} H_z \ . \tag{4.51b}$$

Hence the field structure of an E or H mode can be completely derived from the corresponding longitudinal field component.

4.3 Orthonormality Relations

We turn to an examination of the fundamental physical quantities associated with the electromagnetic field in a waveguide – energy density and energy flux. In the course of the investigation we shall also derive certain orthogonal properties possessed by the electric and magnetic field components of the various modes. Inasmuch as these relations are based on similar orthogonal properties of the scalar functions φ_a and ψ_a, we preface the discussion by a derivation of the necessary theorems. Let us consider two E-mode functions φ_a and φ_b, and construct the identity

$$\int_C \mathrm{d}s \, \varphi_a \frac{\partial}{\partial n} \varphi_b = \int_\sigma \mathrm{d}\sigma \, \boldsymbol{\nabla}_\perp \varphi_a \cdot \boldsymbol{\nabla}_\perp \varphi_b - \gamma_b'^2 \int_\sigma \mathrm{d}\sigma \, \varphi_a \varphi_b \ . \tag{4.52}$$

If we temporarily exclude the T mode of a two-conductor guide, the line integral vanishes by virtue of the boundary condition. On interchanging φ_a and φ_b, and subtracting the resulting equation, we obtain

$$(\gamma_a'^2 - \gamma_b'^2) \int_\sigma \mathrm{d}\sigma \, \varphi_a \varphi_b = 0 \ , \tag{4.53}$$

which demonstrates the orthogonality of two mode functions with different eigenvalues. In consequence of the vanishing of the surface integral in (4.52), we may write this orthogonal relation as

$$\int_\sigma \mathrm{d}\sigma \, \boldsymbol{\nabla}_\perp \varphi_a \cdot \boldsymbol{\nabla}_\perp \varphi_b = 0 \ , \quad \gamma'_a \neq \gamma'_b \ . \tag{4.54}$$

If more than one linearly independent mode function is associated with a particular eigenvalue – a situation which is referred to as “degeneracy” – no guarantee of orthogonality for these eigenfunctions is supplied by (4.53). However, a linear combination of degenerate eigenfunctions is again an eigenfunction, and such linear combinations can always be arranged to have the orthogonal property. In this sense, the orthogonality theorem (4.54) is valid for all pairs of different eigenfunctions. The theorem is also valid for the T mode of a two-conductor system. To prove this, we return to (4.52) and choose the mode a an an ordinary E mode ($\varphi_a = 0$ on C), and the mode b as the T mode ($\gamma'_b = 0$); the desired relation follows immediately. Note, however, that in this situation orthogonality in the form $\int \mathrm{d}\sigma\, \varphi_a \varphi_b = 0$ is not obtained. Finally, then, the orthogonal relation, applicable to all E modes, is

$$\int \mathrm{d}\sigma\, \boldsymbol{\nabla}_\perp \varphi_a \cdot \boldsymbol{\nabla}_\perp \varphi_b = \delta_{ab} \,, \tag{4.55}$$

which also contains a convention regarding the normalization of the E-mode functions:

$$\int \mathrm{d}\sigma\, (\boldsymbol{\nabla}_\perp \varphi)^2 = 1 \,, \tag{4.56}$$

a convenient choice for the subsequent discussion. With the exception of the T mode, the normalization condition can also be written

$$\gamma_a'^2 \int \mathrm{d}\sigma\, \varphi_a^2 = 1 \,. \tag{4.57}$$

The corresponding derivation for H modes proceeds on identical lines, with results expressed by

$$\int \mathrm{d}\sigma\, \boldsymbol{\nabla}_\perp \psi_a \cdot \boldsymbol{\nabla}_\perp \psi_b = \delta_{ab} \,, \tag{4.58}$$

which contains the normalization convention

$$\int \mathrm{d}\sigma\, (\boldsymbol{\nabla}_\perp \psi_a)^2 = \gamma_a''^2 \int \mathrm{d}\sigma\, \psi_a^2 = 1 \,. \tag{4.59}$$

As we shall now see, no statement of orthogonality between E and H modes is required.

4.4 Energy Density and Flux

The energy quantities with which we shall be concerned are the linear energy densities (that is, the energy densities per unit length) obtained by integrating the volume densities across a section of the guide. It is convenient to consider separately the linear densities associated with the various electric and magnetic components of the field. Thus the linear electric energy density connected with the longitudinal electric field is

$$U_{E_z} = \frac{\varepsilon}{2} \int \mathrm{d}\sigma \left[\mathrm{Re} \left(E_z \mathrm{e}^{-\mathrm{i}\omega t} \right) \right]^2 = \frac{\varepsilon}{4} \int \mathrm{d}\sigma\, |E_z|^2 \,, \tag{4.60}$$

where the oscillating terms are omitted due to time-averaging. On inserting the general superposition of individual E-mode fields [cf. (4.32c)],

$$E_z = \frac{\mathrm{i}\zeta}{k} \sum_a \gamma_a'^2 \varphi_a(x,y) I_a'(z) \;, \qquad (4.61)$$

we find

$$U_{E_z} = \frac{\varepsilon}{4} \frac{\zeta^2}{k^2} \sum_a \gamma_a'^2 \left| I_a'(z) \right|^2 \;, \qquad (4.62)$$

in which the orthogonality and normalization of the E-mode functions has been used. The orthogonality of the longitudinal electric fields possessed by different E modes is thus a trivial consequence of the corresponding property of the scalar functions φ_a. The longitudinal electric field energy density can also be written

$$U_{E_z} = \frac{1}{4} \sum_a \frac{1}{\omega^2 C_a'} \left| I_a'(z) \right|^2 \;, \qquad (4.63)$$

by introducing the distributed series capacitance, $C_a' = \varepsilon/\gamma_a'^2$, associated with the transmission line that describes the ath E mode. In a similar way, the linear energy density

$$U_{H_z} = \frac{\mu}{4} \int \mathrm{d}\sigma \, |H_z|^2 \qquad (4.64)$$

derived from the longitudinal magnetic field [(4.33d)]

$$H_z = \frac{\mathrm{i}\eta}{k} \sum_a \gamma_a''^2 \psi_a(x,y) V_z''(z) \qquad (4.65)$$

reads

$$U_{H_z} = \frac{\mu}{4} \frac{\eta^2}{k^2} \sum_a \gamma_a''^2 \left| V_a''(z) \right|^2 \;, \qquad (4.66)$$

in consequence of the normalization condition for ψ_a and the orthogonality of the longitudinal magnetic fields of different H modes. The insertion of the distributed shunt inductance characteristic of the ath H-mode transmission line, $L_a'' = \mu/\gamma_a''^2$, transforms this energy density expression into

$$U_{H_z} = \frac{1}{4} \sum_a \frac{1}{\omega^2 L_a''} \left| V_a''(z) \right|^2 \;. \qquad (4.67)$$

To evaluate the linear energy density associated with the transverse electric field

$$U_{\mathbf{E}_\perp} = \frac{\varepsilon}{4} \int \mathrm{d}\sigma \, |\mathbf{E}_\perp|^2 \;, \qquad (4.68)$$

it is convenient to first insert the general representation (4.8), thus obtaining

$$U_{\mathbf{E}_\perp} = \frac{\varepsilon}{4} \left[\int \mathrm{d}\sigma \, |\boldsymbol{\nabla}_\perp V'|^2 + \int \mathrm{d}\sigma \, |\mathbf{e} \times \boldsymbol{\nabla} V''|^2 - 2\mathrm{Re} \int \mathrm{d}\sigma \, \boldsymbol{\nabla}_\perp V' \cdot \mathbf{e} \times \boldsymbol{\nabla} V''^* \right] . \qquad (4.69)$$

The last terms of this expression, representing the mutual energy of the E and H modes, may be proved to vanish by the following sequence of equations:

$$\begin{aligned}\int_\sigma \mathrm{d}\sigma\, \boldsymbol{\nabla}_\perp V' \cdot \mathbf{e} \times \boldsymbol{\nabla} V''^* &= -\int_\sigma \mathrm{d}\sigma\, \boldsymbol{\nabla}_\perp V''^* \cdot \mathbf{e} \times \boldsymbol{\nabla} V' \\ &= -\int_\sigma \mathrm{d}\sigma\, \boldsymbol{\nabla}_\perp \cdot (V''^* \mathbf{e} \times \boldsymbol{\nabla} V') \\ &= -\int_C \mathrm{d}s\, V''^* \mathbf{n} \cdot \mathbf{e} \times \boldsymbol{\nabla} V' = \int_C \mathrm{d}s\, V''^* \frac{\partial}{\partial s} V' = 0\,,\end{aligned} \tag{4.70}$$

in which the last step involves the generally valid boundary condition, $\frac{\partial}{\partial s} V' = 0$ on C, see (4.21). (A proof employing the boundary condition $V' = 0$ on C would not apply to the T mode.) It has thus been shown that the transverse electric field of an E mode is orthogonal to the transverse electric field of an H mode. For the transverse electric field energy density of the E mode, we have from (4.14a)

$$\frac{\varepsilon}{4} \int \mathrm{d}\sigma\, |\boldsymbol{\nabla}_\perp V'|^2 = \frac{\varepsilon}{4} \sum_a |V'_a(z)|^2\,, \tag{4.71}$$

as an immediate consequence of the orthonormality condition (4.55), which demonstrates the orthogonality of the transverse electric fields of different E modes. Similarly, from (4.17a) the transverse electric field energy density of the H modes:

$$\frac{\varepsilon}{4} \int \mathrm{d}\sigma\, |\mathbf{e} \times \boldsymbol{\nabla} V''|^2 = \frac{\varepsilon}{4} \int \mathrm{d}\sigma\, |\boldsymbol{\nabla}_\perp V''|^2 = \frac{\varepsilon}{4} \sum_a |V''(z)|^2\,, \tag{4.72}$$

is a sum of individual mode contributions, indicating the orthogonality of the transverse electric fields of different H modes. Finally, the transverse electric field energy density is

$$U_{\mathbf{E}_\perp} = \frac{1}{4} \sum_a C |V'_a(z)|^2 + \frac{1}{4} \sum_a C |V''_a(z)|^2\,, \tag{4.73}$$

where $C = \varepsilon$ is the distributed shunt capacitance common to all E- and H-mode transmission lines.

The discussion of the transverse magnetic field energy density,

$$U_{\mathbf{H}_\perp} = \frac{\mu}{4} \int \mathrm{d}\sigma\, |\mathbf{H}_\perp|^2 = \frac{\mu}{4} \int \mathrm{d}\sigma\, |\mathbf{H} \times \mathbf{e}|^2\,, \tag{4.74}$$

is precisely analogous and requires no detailed treatment, for in virtue of (4.8) and (4.9a) it is merely necessary to make the substitutions $V' \to I'$, $V'' \to I''$ (and $\varepsilon \to \mu$, of course) to obtain the desired result. The boundary condition upon which the analogue of (4.70) depends now reads $\frac{\partial}{\partial s}I' = 0$ on C, which is again an expression of the E-mode boundary condition. Hence

$$U_{\mathbf{H}_\perp} = \frac{1}{4}\sum_a L|I'_a(z)|^2 + \frac{1}{4}\sum_a L|I''_a(z)|^2 \,, \tag{4.75}$$

where $L = \mu$ is the distributed series inductance characteristic of all mode transmission lines. The orthogonality of the transverse magnetic fields associated with two different modes, which is contained in the result, may also be derived from the previously established transverse electric field orthogonality with the aid of the relations between transverse field components that is exhibited in (4.45a), (4.45b).

The complex power flowing along the waveguide is obtained from the longitudinal component of the complex Poynting vector by integration across a guide section:

$$P = \frac{1}{2}\int \mathrm{d}\sigma\, \mathbf{E}\times\mathbf{H}^*\cdot\mathbf{e} = \frac{1}{2}\int \mathrm{d}\sigma\, \mathbf{E}\cdot(\mathbf{H}\times\mathbf{e})^* \,, \tag{4.76}$$

whence from (4.8) and (4.9b)

$$\begin{aligned} P &= \frac{1}{2}\bigg[\int \mathrm{d}\sigma\, \boldsymbol{\nabla}_\perp V'\cdot\boldsymbol{\nabla}_\perp I'^* + \int \mathrm{d}\sigma\, \mathbf{e}\times\boldsymbol{\nabla}V''\cdot\mathbf{e}\times\boldsymbol{\nabla}I''^* \\ &\qquad - \int \mathrm{d}\sigma\, \boldsymbol{\nabla}V'\cdot\mathbf{e}\times\boldsymbol{\nabla}I''^* - \int \mathrm{d}\sigma\, \mathbf{e}\times\boldsymbol{\nabla}V''\cdot\boldsymbol{\nabla}I'^*\bigg] \\ &= \frac{1}{2}\sum_a V'_a(z)I'_a(z)^* + \frac{1}{2}\sum_a V''_a(z)I''_a(z)^* \,, \end{aligned} \tag{4.77}$$

which uses (4.14a), (4.14b) and (4.17a), (4.17b), and the analogue of (4.70). Hence the complex power flow is a sum of individual mode contributions, each having the proper transmission-line form; it will now be evident that the normalization conditions (4.56) and (4.59) were adopted in anticipation of this result. The orthogonality that is implied by expression (4.77) is a simple consequence of the orthogonality property of transverse electric fields, since $\mathbf{H}\times\mathbf{e}$ for an individual mode is proportional to the corresponding transverse electric field.

It will have been noticed that the linear energy densities associated with the different field components are in full agreement with the energies stored per unit length in the various elements of the distributed parameter circuits. Thus the E- and H-type circuits give a complete pictorial description of the electromagnetic properties of E and H modes in the usual sense: Capacitance and inductance represents electric and magnetic energy; series elements are associated with longitudinal electric and transverse magnetic fields (longitudinal displacement and conduction currents); shunt elements describe transverse electric and longitudinal magnetic fields (transverse displacement and conduction currents). The air of precise definition attached to the line parameters, however, is spurious. We are at liberty to multiply a transmission line voltage by a constant and divide the associated current by the same constant without violating the requirement that the complex power have the transmission line form. Thus, let a mode voltage and current be replaced by $N^{-1/2}V(z)$ and $N^{1/2}I(z)$, respectively, implying that the new voltage and current are obtained from the old definitions through multiplication by $N^{1/2}$ and $N^{-1/2}$, respectively. In order the preserve the form of the energy expressions, the inductance parameters must be multiplied by N, and the capacitance parameters divided by N. It follows from these statements that the characteristic impedance must by multiplied by N, in agreement with its significance as a voltage-current ratio. The propagation constant is unaffected by this alteration, of course. We may conclude that one of the basic quantities that specifies the transmission line, characteristic impedance, remains essentially undefined by any considerations thus far introduced. The same situation arose in the field analysis of the two-conductor transmission line and it was shown that a natural definition for the characteristic impedance could be obtained by ascribing the customary physical meaning to either the current or voltage, the same result being obtained in either event. This somewhat artificial procedure was employed in order to emphasize the rather different character of waveguide fields for, as we shall now show, a precise definition of characteristic impedance can be obtained by ascribing a physical significance to either the current or the voltage, depending on the type of mode, but not to both simultaneously.

An E mode is essentially characterized by E_z, from which all other field components can be derived. Associated with the longitudinal electric field is an electric displacement current, the current density being [(4.32c)]

$$-\mathrm{i}\omega\varepsilon E_z = \gamma^2\varphi(x,y)N^{1/2}I(z) \ . \tag{4.78}$$

In addition, there is a longitudinal electric conduction current on the metal walls, with the surface density, according to (4.32b)

$$-(\mathbf{n}\times\mathbf{H})_z = \frac{\partial}{\partial n}\varphi(x,y)N^{1/2}I(z) \ , \quad (x,y)\in S \ . \tag{4.79}$$

The total (conduction plus displacement) longitudinal electric current is zero,[3] and it is natural to identify the total current flowing in the positive direction with $I(z)$, which leads to the following equation for N:

$$N^{1/2}\left[\gamma^2\int_+ \mathrm{d}\sigma\,\varphi + \int_+ \mathrm{d}s\,\frac{\partial}{\partial n}\varphi\right] = 1 \ , \tag{4.80}$$

[3] This follows from the fact that $\mathbf{H}=\mathbf{0}$ in the conductor, so that $\int_C \mathrm{d}\mathbf{s}\cdot\mathbf{H} = \int_\sigma \mathrm{d}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\times\mathbf{H} = 0$.

where the surface and line integrals, denoted by $\int_+$, are to be conducted over those regions where φ and $\frac{\partial}{\partial n}\varphi$ are positive. This equation is particularly simple for the lowest E mode in any hollow waveguide, that is, the mode of minimum cutoff frequency, for this mode has the property, to be established in Sec. **??**, that the scalar function φ is nowhere negative, and vanishes only on the boundary. It follows that the normal derivative on the boundary cannot be positive. Hence the displacement current flows entirely in the positive direction, and the conduction current entirely in the negative direction. Consequently,

$$N = \frac{1}{\gamma^4 \left(\int \mathrm{d}\sigma\, \varphi\right)^2} = \frac{1}{\gamma^2} \frac{\int \mathrm{d}\sigma\, \varphi^2}{\left(\int \mathrm{d}\sigma\, \varphi\right)^2} \,, \tag{4.81}$$

on employing the normalization condition for φ, (4.57), to express N in a form that is independent of the absolute scale of the function φ, Therefore, for the lowest E mode in any guide, a natural choice of characteristic impedance is, from (4.36b),

$$Z = \zeta \frac{\kappa}{k} \frac{1}{\gamma^2} \frac{\int \mathrm{d}\sigma\, \varphi^2}{\left(\int \mathrm{d}\sigma\, \varphi\right)^2} \,. \tag{4.82}$$

For the other E modes, the ϕ normalization condition can be used in an analogous way to obtain

$$Z = \zeta \frac{\kappa}{k} \frac{1}{\gamma^2} \frac{\int \mathrm{d}\sigma\, \varphi^2}{\left(\int_+ \mathrm{d}\sigma\, \varphi + \frac{1}{\gamma^2} \int_+ \mathrm{d}s\, \frac{\partial}{\partial n}\varphi\right)^2} \,. \tag{4.83}$$

It may appear more natural to deal with the voltage rather than the current in the search for a proper characteristic impedance definition, since the transverse electric field of an E mode is derived from a potential [(4.32a)]. The voltage could then be defined as the potential of some fixed point with respect to the wall in a given cross section, thus determining N. In a guide of symmetrical cross section the only natural reference point is the center, which entails the difficulty that there exists an infinite class of modes for which $\varphi = 0$ at the center, and the definition fails. In addition, when this does not occur, as in the lowest E mode, the potential of the center point does not necessarily equal the voltage, if the characteristic impedance is defined on a current basis as we have done. Hence, while significance can always be attached to the E-mode current, no generally valid voltage definition can be offered.

By analogy with the E-mode discussion, we shall base a characteristic admittance definition for H modes on the properties of H_z, which can be said to define a longitudinal magnetic displacement current density, from (4.33d)

$$-\mathrm{i}\omega\mu H_z = \gamma^2 \psi(x, y) N^{-1/2} V(z) \,. \tag{4.84}$$

The total longitudinal magnetic displacement current is zero[4] and we shall identify the total magnetic current flowing in the positive direction with $V(z)$.

[4] Because $\mathbf{E} = \mathbf{0}$ in the conductor, so is $\int_C \mathrm{d}\mathbf{s} \cdot \mathbf{E} = \int_\sigma \mathrm{d}\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \times \mathbf{E}$.

The voltage thus defined equals the line integral of the electric field intensity taken clockwise around all regions through which positive magnetic displacement flows. Accordingly,

$$N = \gamma^4 \left(\int_+ \mathrm{d}\sigma \, \psi \right)^2 = \gamma^2 \frac{\left(\int_+ \mathrm{d}\sigma \, \psi \right)^2}{\int \mathrm{d}\sigma \, \psi^2} \tag{4.85}$$

and [cf. (4.36d)]

$$Y = \eta \frac{\kappa}{k} \frac{1}{\gamma^2} \frac{\int \mathrm{d}\sigma \, \psi^2}{\left(\int_+ \mathrm{d}\sigma \, \psi \right)^2} \, . \tag{4.86}$$

It would also be possible to base an admittance definition on the identification of the transmission-line current with the total longitudinal electric conduction current flowing in the positive direction on the metal walls. However, the characteristic admittance so obtained will not agree in general with that just obtained.

Although we have advanced rather reasonable definitions of characteristic impedance and admittance, it is clear that these choices possess arbitrary features and in no sense can be considered inevitable. This statement may convey the impression that the theory under development is essentially vague and ill-defined, which would be a misunderstanding. Physically observable quantities can in no way depend on the precise definition of a characteristic impedance, but this does not detract from its appearance in a theory which seeks to express its results in conventional circuit language. Indeed, the arbitrariness in definition is a direct expression of the greater complexity of waveguide systems compared with low-frequency transmission lines. For example, in the junction of two low-frequency transmission lines with different dimensions, the conventional transmission-line currents and voltages are continuous to a high degree of approximation and hence the reflection properties of the junction are completely specified by the quantities which relate the current and voltage in each line. In a corresponding waveguide situation, however, physical quantities with such simple continuity properties do not exist in general, and it is therefore not possible to describe the properties of the junction in terms of two quantities which are each characteristic of an individual guide. Armed with this knowledge, which anticipates the results to be obtained in subsequent chapters, we are forced to the position that the characteristic impedance is best regarded as a quantity chosen to simplify the electrical representation of particular situation, and that different definitions may be advantageously employed in different circumstances. In particular, the impedance definition implicitly adopted at the beginning of the chapter, corresponding to $N = 1$, is most convenient for general theoretical discussion since it directly relates the transverse electric and magnetic fields. This choice, which may be termed the field impedance (admittance), will be adhered to in the remainder of this chapter.

Before turning to the discussion of particular types of guides, we shall derive a few simple properties of the electric and magnetic energies associated with propagating and nonpropagating modes. The tools for the purpose are provided by the complex Poynting vector theorem (ε real)

$$\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}^*) = \mathrm{i}\omega \left(\mu |\mathbf{H}|^2 - \varepsilon |\mathbf{E}|^2\right) , \tag{4.87}$$

and the energy theorem (ignoring any dependence of ε and μ on the frequency)

$$\boldsymbol{\nabla} \cdot \left(\frac{\partial \mathbf{E}}{\partial \omega} \times \mathbf{H}^* + \mathbf{E}^* \times \frac{\partial \mathbf{H}}{\partial \omega} \right) = \mathrm{i} \left(\varepsilon |\mathbf{E}|^2 + \mu |\mathbf{H}|^2 \right) . \tag{4.88}$$

[Proofs and generalizations of these theorems are given in the Problems at the end of this chapter.] If (4.87) is integrated over a cross section of the guide, only the longitudinal component of the Poynting vector survives (because $\mathbf{E}_\perp$ vanishes on S), and we obtain the transmission-line form of the complex Poynting vector theorem, as applied to a single mode [cf. (4.77)]:

$$\frac{\mathrm{d}}{\mathrm{d}z} \left[\frac{1}{2} V(z) I(z)^* \right] = \frac{\mathrm{d}}{\mathrm{d}z} P = 2\mathrm{i}\omega (U_H - U_E) , \tag{4.89}$$

where U_E and U_H are the electric and magnetic linear energy densities. A similar operation on (4.88) yields

$$\frac{\mathrm{d}}{\mathrm{d}z} \left\{ \frac{1}{2} \left[\frac{\partial V(z)}{\partial \omega} I(z)^* + V^*(z) \frac{\partial I(z)}{\partial \omega} \right] \right\} = 2\mathrm{i}(U_E + U_H) = 2\mathrm{i}U , \tag{4.90}$$

since $\partial \mathbf{E}_t / \partial \omega$, for example, involves $\partial V(z)/\partial \omega$ in the same way that $\mathbf{E}_t$ contains $V(z)$, for the scalar mode functions do not depend upon the frequency. As a first application of these equations we consider a propagating wave progressing in the positive direction, i.e.,

$$\begin{aligned} V(z) &= V \mathrm{e}^{\mathrm{i}\kappa z} , \\ I(z) &= I \mathrm{e}^{\mathrm{i}\kappa z} , \end{aligned} \quad V = ZI . \tag{4.91}$$

The complex power is real and independent of z:

$$P = \frac{1}{2} V I^* = \frac{1}{2} Z |I|^2 , \tag{4.92}$$

whence we deduce from (4.89) that $U_E = U_H$; the electric and magnetic linear energy densities are equal in a progressive wave. To apply the energy theorem we observe that

$$\frac{\partial V(z)}{\partial \omega} = \frac{\partial V}{\partial \omega} \mathrm{e}^{\mathrm{i}\kappa z} + \mathrm{i} \frac{\mathrm{d}\kappa}{\mathrm{d}\omega} z V \mathrm{e}^{\mathrm{i}\kappa z} , \tag{4.93a}$$

$$\frac{\partial I(z)}{\partial \omega} = \frac{\partial I}{\partial \omega} \mathrm{e}^{\mathrm{i}\kappa z} + \mathrm{i} \frac{\mathrm{d}\kappa}{\mathrm{d}\omega} z I \mathrm{e}^{\mathrm{i}\kappa z} , \tag{4.93b}$$

and that

$$\frac{1}{2}\left[\frac{\partial V(z)}{\partial\omega}I(z)^* + V^*(z)\frac{\partial I(z)}{\partial\omega}\right] = \frac{1}{2}\left(\frac{\partial V}{\partial\omega}I^* + V^*\frac{\partial I}{\partial\omega}\right) + \mathrm{i}\frac{\mathrm{d}\kappa}{\mathrm{d}\omega}z\frac{1}{2}(VI^* + V^*I) \; . \tag{4.94}$$

Therefore, (4.90) implies

$$\frac{\mathrm{d}\kappa}{\mathrm{d}\omega}\frac{1}{4}(VI^* + V^*I) = \frac{\mathrm{d}\kappa}{\mathrm{d}\omega}P = U \; , \tag{4.95}$$

or

$$P = vU \; , \tag{4.96}$$

where

$$v = \frac{\mathrm{d}\omega}{\mathrm{d}\kappa} = c\frac{\mathrm{d}k}{\mathrm{d}\kappa} = c\frac{\kappa}{k} \; . \tag{4.97}$$

The relation thus obtained expresses a proportionality between the power transported by a progressive wave and the linear energy density. The coefficient v must then be interpreted as the velocity of energy transport. It is consistent with this interpretation that v is always less than c, and vanishes at the cutoff frequency. At frequencies large in comparison with the cutoff frequency, v approaches the intrinsic velocity of the medium. It is interesting to compare this velocity with the two velocities already introduced in discussing one-dimensional propagation in a dispersive medium – the phase and group velocities. The phase velocity equals the ratio of the angular frequency and the propagation constant:

$$u = \frac{\omega}{\kappa} = c\frac{k}{\kappa} \; , \tag{4.98}$$

while the group velocity is the derivative of the angular frequency with respect to the propagation constant:

$$v = \frac{\mathrm{d}\omega}{\mathrm{d}\kappa} \; . \tag{4.99}$$

That the group velocity and energy transport velocity are equal is not unexpected. We notice that the phase velocity always exceeds the intrinsic velocity of the medium, and indeed is infinite at the cutoff frequency of the mode. The two velocities are related by

$$uv = c^2 \tag{4.100}$$

which is reminiscent of (1.39). A simple physical picture for the phase and group velocities will be offered in Chapter 5.

Another derivation of the energy transport velocity, which makes more explicit use of the waveguide fields, is suggested by the defining equation:

$$v = \frac{P}{U} = \frac{\int \mathrm{d}\sigma\, \mathbf{e}\cdot\mathbf{E}\times\mathbf{H}^*}{\frac{1}{2}\int \mathrm{d}\sigma\,(\varepsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2)} \; . \tag{4.101}$$

In virtue of the equality of electric and magnetic linear energy densities, and the relation $\mathbf{e} \times \mathbf{E} = Z\mathbf{H}$, which is valid for an E-mode field propagating in the positive direction [(4.45a)], we find using (4.36b)

$$v = \frac{Z}{\mu} \frac{\int \mathrm{d}\sigma \, |\mathbf{H}_\perp|^2}{\int \mathrm{d}\sigma \, |\mathbf{H}|^2} = c\frac{\kappa}{k} \,, \tag{4.102}$$

since $\mathbf{H}$ has no longitudinal component. Similarly, the energy transport velocity for an H mode is from (4.45b) and (4.36d)

$$v = \frac{Y}{\varepsilon} \frac{\int \mathrm{d}\sigma \, |\mathbf{E}_\perp|^2}{\int \mathrm{d}\sigma \, |\mathbf{E}|^2} = c\frac{\kappa}{k} \,. \tag{4.103}$$

When the wave motion on the transmission line is not that of a simple progressive wave, but the general superposition of standing waves (or running waves) described by

$$V(z) = V \cos \kappa z + \mathrm{i} Z I \sin \kappa z \,, \tag{4.104a}$$

$$I(z) = I \cos \kappa z + \mathrm{i} Y V \sin \kappa z \,, \tag{4.104b}$$

or

$$V(z) = (2Z)^{1/2} \left(A \mathrm{e}^{\mathrm{i}\kappa z} + B \mathrm{e}^{-\mathrm{i}\kappa z} \right) \,, \tag{4.105a}$$

$$I(z) = (2Y)^{1/2} \left(A \mathrm{e}^{\mathrm{i}\kappa z} - B \mathrm{e}^{-\mathrm{i}\kappa z} \right) \,, \tag{4.105b}$$

the electric and magnetic linear energy densities are not equal, in general, since the complex power is a function of position on the line:

$$P(z) = \frac{1}{2} \left[V I^* \cos^2 \kappa z + V^* I \sin^2 \kappa z + \mathrm{i} \sin \kappa z \cos \kappa z \left(Z|I|^2 - Y|V|^2 \right) \right] \,, \tag{4.106}$$

or

$$P(z) = |A|^2 - |B|^2 - A B^* \mathrm{e}^{2\mathrm{i}\kappa z} + A^* B \mathrm{e}^{-2\mathrm{i}\kappa z} \,. \tag{4.107}$$

However, equality is obtained for the total electric and magnetic energies stored in any length of line that is an integral multiple of $\frac{1}{2}\lambda_g$. To prove this, we observe that by integrating (4.89)

$$P(z_2) - P(z_1) = 2\mathrm{i}\omega (W_H - W_E) \,, \tag{4.108}$$

where W_E and W_H are the total electric and magnetic energies stored in the length of transmission line between the points z_1 and z_2. Now, the complex power is a periodic function of z with the periodicity interval $\pi/\kappa = \frac{1}{2}\lambda_g$ [(4.39)], from which we conclude that $P(z_2) = P(z_1)$ if the two points are separated by an integral number of half guide wavelengths, which verifies the statement. An equivalent form of this result is that the average electric and magnetic energy densities are equal, providing the averaging process is extended over an integral number of half-guide wavelengths, or over a distance large in comparison with $\frac{1}{2}\lambda_g$.

An explicit expression for the average energy density can be obtained from the energy theorem (4.88). The total energy W, stored in the guide between the planes $z = z_1$ and $z = z_2$, is given by the integral of (4.90), or

$$W = \frac{1}{4\mathrm{i}} \left[\frac{\partial V(z)}{\partial \omega} I(z)^* + V(z)^* \frac{\partial I(z)}{\partial \omega} \right]_{z=z_1}^{z=z_2} . \tag{4.109}$$

On differentiating the voltage and current expressions (4.104a), (4.104b) with respect to the frequency, we find

$$\frac{\partial V(z)}{\partial \omega} = \mathrm{i}z \frac{\mathrm{d}\kappa}{\mathrm{d}\omega} Z I(z) + \left[\cos \kappa z \frac{\partial V}{\partial \omega} + \mathrm{i} \sin \kappa z \frac{\partial Z I}{\partial \omega} \right] , \tag{4.110a}$$

$$\frac{\partial I(z)}{\partial \omega} = \mathrm{i}z \frac{\mathrm{d}\kappa}{\mathrm{d}\omega} Y V(z) + \left[\cos \kappa z \frac{\partial I}{\partial \omega} + \mathrm{i} \sin \kappa z \frac{\partial Y V}{\partial \omega} \right] . \tag{4.110b}$$

Hence

$$\frac{\partial V(z)}{\partial \omega} I(z)^* + V(z)^* \frac{\partial I(z)}{\partial \omega} = \mathrm{i}z \frac{\mathrm{d}\kappa}{\mathrm{d}\omega} \left[Z|I(z)|^2 + Y|V(z)|^2 \right] + \dots , \tag{4.111}$$

where the unwritten part of this equation consists of those terms, arising from the bracketed expressions in (4.110a) and (4.110b), which are periodic functions of z with the period $\frac{1}{2}\lambda_g$. Thus, if the points z_1 and z_2 are separated by a distance that is an integral multiple of $\frac{1}{2}\lambda_g$, these terms make no contribution to the total energy. We also note that the quantity $Z|I(z)|^2 + Y|V(z)|^2$ is independent of z:

$$Z|I(z)|^2 + Y|V(z)|^2 = Z|I|^2 + Y|V|^2 = 4(|A|^2 + |B|^2) , \tag{4.112}$$

from which we conclude that the total energy stored in a length of guide, l, which is an integral number of half-guide wavelengths, is, in terms of the energy velocity (4.97)

$$W = l \frac{\mathrm{d}\kappa}{\mathrm{d}\omega} \frac{1}{4} \left(Z|I|^2 + Y|V|^2 \right) = \frac{l}{v} \frac{1}{4} \left(Z|I|^2 + Y|V|^2 \right) = \frac{l}{v} \left(|A|^2 + |B|^2 \right) . \tag{4.113}$$

The average total energy density is W/l, which has a simple physical significance in terms of running waves, being just the sum of the energy densities associated with each progressive wave component if it alone existed on the transmission line.

The energy relations for a nonpropagating mode are rather different; there is a definite excess of electric or magnetic energy, depending on the type of mode. The propagation constant for a nonpropagating (below cutoff) mode is imaginary:

$$\kappa = \mathrm{i}\sqrt{\gamma^2 - k^2} = \mathrm{i}|\kappa| , \tag{4.114}$$

and a field that is attenuating in the positive z-direction is described by

$$\begin{aligned} V(z) &= V\mathrm{e}^{-|\kappa|z} \\ I(z) &= I\mathrm{e}^{-|\kappa|z} \end{aligned} \qquad V = ZI \; . \tag{4.115}$$

The imaginary characteristic impedance (admittance) of an E(H) mode is given by

$$\text{E mode:} \quad Z = \mathrm{i}\zeta\frac{|\kappa|}{k} = \mathrm{i}|Z| \; , \tag{4.116a}$$

$$\text{H mode:} \quad Y = \mathrm{i}\eta\frac{|\kappa|}{k} = \mathrm{i}|Y| \; . \tag{4.116b}$$

The energy quantities of interest are the total electric and magnetic energy stored in the positive half of the guide ($z > 0$). The difference of these energies is given by (4.108), where $z_1 = 0$ and $z_2 \to \infty$. Since all field quantities approach zero exponentially for increasing z, $P(z_2) \to 0$, and

$$W_E - W_H = \frac{1}{2\mathrm{i}\omega}P(0) = \frac{1}{4\mathrm{i}\omega}VI^* \; . \tag{4.117}$$

For an E mode

$$VI^* = Z|I|^2 = \mathrm{i}|Z||I|^2 \; , \tag{4.118}$$

and so

$$\text{E mode:} \quad W_E - W_H = \frac{1}{4\omega}|Z||I|^2 \; , \tag{4.119}$$

which is positive. Hence an E mode below cutoff has an excess of electric energy, in agreement with the capacitive reactance form (see below) of the characteristic impedance. Similarly, for an H mode,

$$VI^* = Y^*|V|^2 = -\mathrm{i}|Y||V|^2 \; , \tag{4.120}$$

whence

$$\text{H mode:} \quad W_H - W_E = \frac{1}{4\omega}|Y||V|^2 \; , \tag{4.121}$$

implying that an H mode below cutoff preponderantly stores magnetic energy, as the inductive susceptance form (see below) of its characteristic admittance would suggest.

To obtain the total energy stored in a nonpropagating mode, we employ (4.109), again with $z_1 = 0$ and $z_2 \to \infty$:

$$W = \frac{\mathrm{i}}{4}\left(\frac{\partial V}{\partial \omega}I^* + V^*\frac{\partial I}{\partial \omega}\right) \; . \tag{4.122}$$

On differentiating the relation $V = ZI$ with respect to ω, and making appropriate substitutions in (4.122), we find

$$W = \frac{\mathrm{i}}{4}\left[\frac{\mathrm{d}Z}{\mathrm{d}\omega}|I|^2 + (Z + Z^*)\frac{\partial I}{\partial \omega}I^*\right] \; . \tag{4.123}$$

The imaginary form of Z (= i$|Z|$ for an E mode) then implies that

$$W = -\frac{1}{4}\frac{\mathrm{d}|Z|}{\mathrm{d}\omega}|I|^2 \ . \tag{4.124}$$

We may note in passing that the positive nature of the total energy demands that $|Z|$ be a decreasing function of frequency, or better, that the reactance characterizing Z $(= -\mathrm{i}X)$ be an increasing function of frequency. The requirement is verified by direct differentiation:

$$-\frac{\mathrm{d}|Z|}{\mathrm{d}\omega} = \frac{1}{\omega}\frac{\gamma^2}{\gamma^2 - k^2}|Z| \ , \tag{4.125}$$

and

$$\text{E mode:} \quad W = \frac{1}{4\omega}\frac{\gamma^2}{\gamma^2 - k^2}|Z||I|^2 \ . \tag{4.126}$$

A comparison of this result with (4.119) shows that

$$\frac{2W_H}{W} = \frac{k^2}{\gamma^2} \ . \tag{4.127}$$

Thus, the electric and magnetic energies are equal just at the cutoff frequency $(k = \gamma)$, and as the frequency diminishes, the magnetic energy steadily decreases in comparison with the electric energy. In conformity with the latter remark, the E-mode characteristic impedance approaches $\mathrm{i}\zeta\gamma/k = \mathrm{i}\gamma/\omega\epsilon$ when $k/\gamma \ll 1$, which implies that a transmission line describing an attenuated E mode at a frequency considerably below the cutoff frequency behaves like a lumped capacitance $C = \varepsilon/\gamma = \varepsilon\lambda_c/(2\pi)$.

The H-mode discussion is completely analogous, with the roles of electric and magnetic fields interchanged. Thus the total stored energy is

$$W = -\frac{1}{4}\frac{\mathrm{d}|Y|}{\mathrm{d}\omega}|V|^2 \ , \tag{4.128}$$

implying that the susceptance characterizing Y $(= -\mathrm{i}B)$ must be an increasing function of frequency. Explicitly,

$$\text{H mode:} \quad W = \frac{1}{4\omega}\frac{\gamma^2}{\gamma^2 - \omega^2}|Y||V|^2 \ , \tag{4.129}$$

and

$$\frac{2W_E}{W} = \frac{k^2}{\gamma^2} \ . \tag{4.130}$$

At frequencies well below the cutoff frequency, the electric energy is negligible in comparison with the magnetic energy, and the characteristic admittance becomes $\mathrm{i}\eta\gamma/k = \mathrm{i}\gamma/\omega\mu$. Thus, an H-mode transmission line under these circumstances behaves like a lumped inductance $L = \mu/\gamma = \mu\lambda_c/(2\pi)$.

4.5 Problems for Chapter 4

1. Prove the complex Poynting vector theorem, (4.87), and the energy theorem, (4.88), starting from the definitions of the Fourier transforms in time:

$$\mathbf{E}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t}\mathbf{E}(t) \,, \tag{4.131a}$$

$$\mathbf{H}^*(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t}\mathbf{H}(t) \,. \tag{4.131b}$$

What are the general forms of these theorems if no connection is assumed between $\mathbf{D}(\omega)$ and $\mathbf{E}(\omega)$ and between $\mathbf{B}(\omega)$ and $\mathbf{H}(\omega)$?

2. Show that if dispersion be included, the generalization of (4.88) is

$$\begin{aligned}&\boldsymbol{\nabla}\cdot\left(\frac{\partial\mathbf{E}(\omega)}{\partial\omega}\times\mathbf{H}^*(\omega)+\mathbf{E}^*\times\frac{\partial\mathbf{H}(\omega)}{\partial\omega}\right)\\ &=\mathrm{i}\left[\left(\frac{\mathrm{d}}{\mathrm{d}\omega}(\omega\varepsilon)\right)|\mathbf{E}|^2+\left(\frac{\mathrm{d}}{\mathrm{d}\omega}(\omega\mu)\right)|\mathbf{H}|^2\right]\equiv 4\mathrm{i}\tilde{w}\,.\end{aligned} \tag{4.132}$$

3. Calculate the corresponding group velocity v, defined as the ratio of the rate of energy flow or power

$$P = \frac{1}{2}\int_\sigma \mathrm{d}\sigma\,\mathbf{E}\times\mathbf{H}^*\cdot\hat{\mathbf{e}}, \tag{4.133}$$

where $\hat{\mathbf{e}}$ is the direction of propagation of the electromagnetic disurbance and the integration is over the corresponding perpendicular area σ, to the energy per unit length,

$$\tilde{U} = \int_\sigma \mathrm{d}\sigma\,\tilde{w}\,. \tag{4.134}$$

Assuming that the time averaged electric and magnetic energies per unit length are equal, show that

$$v = \frac{c}{1-\frac{\mathrm{d}\ln c}{\mathrm{d}\ln\omega}}\,, \tag{4.135}$$

where c is the speed of light in the medium. Calculate this in the example of the plasma model, where $\varepsilon = \varepsilon_0(1-\omega_p^2/\omega^2)$, $\mu=\mu_0$, in terms of the parameter called the plasma frequency ω_p, and show that $v < c$.

4.6 Appendix: Two-Conductor Transmission Line

An ideal two-wire transmission line can be thought of as a series of elements, each of which consists of a series inductance L_s and capacitance C_s, and a shunt inductance $L_\perp$ and capacitance $C_\perp$, as illustrated in Fig. 4.1. Let the length of each element be Δz. Then the voltage drop across the element is, for a given frequency ω,

$$\Delta V = \mathrm{i}\omega L_s I + \frac{1}{\mathrm{i}\omega C_s} I \ , \tag{4.136}$$

from which we infer a series impedance per unit length

$$Z_s = \mathrm{i}\omega L_s + \frac{1}{\mathrm{i}\omega C_s} \ . \tag{4.137}$$

Similarly, because the current shorted between the two conductors is

$$\Delta I = \mathrm{i}\omega_\perp V + \frac{1}{\mathrm{i}\omega C_\perp} V \ , \tag{4.138}$$

the shunt admittance per unit length is

$$Y_\perp = \mathrm{i}\omega_\perp V + \frac{1}{\mathrm{i}\omega C_\perp} \ . \tag{4.139}$$

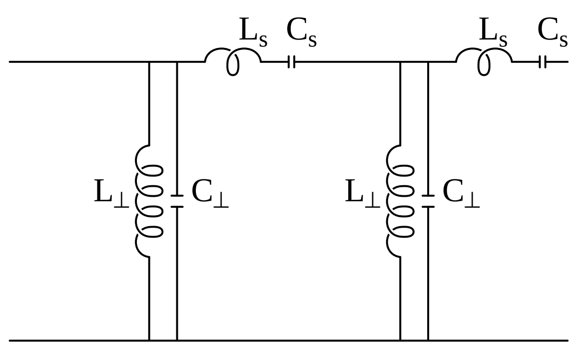

Fig. 4.1. Transmission line represented in terms of equivalent series and shunt inductances and capacitances. Represented here are two elements, each of length Δz, which are repeated indefinitely.