Chapter 4

Macroscopic Electrodynamics

4.1 Force on an Atom

The Maxwell-Lorentz system of equations, (1.65) and (1.68), provides a microscopic description of electromagnetic phenomena, at the classical level, ranging from the simplest two-particle system to the detailed behavior of all particles in a macroscopic system. However, for the latter case, we usually do not require such a complete description, since our measurements involve macroscopic quantities which are only indirectly related to the microscopic behavior of individual atoms. We must develop a theory that is directly applicable to the macroscopic situation with only an implicit reference back to the detailed characterization of the system. The resulting macroscopic electrodynamics is a *phenomenological* theory, by which is meant a theory that operates at the level of the phenomena being correlated and predicted, while maintaining the possibility of contact with a more fundamental theory—here, microscopic electrodynamics—that operates at a deeper level. That contact exists to the extent that the macroscopic measurements can be considered to be averages, over very many atoms, of the results of hypothetical microscopic measurements.

To begin, we consider an atom, an electrically neutral assembly of point charges,

$$\sum_a e_a = 0, \tag{4.1}$$

that are bound together in a small region. We want to study the response of such a system to external electric and magnetic fields that vary only slightly over the spatial extent of that system. We will first concentrate our attention on the net force on the system at a given time, the sum of the forces on its constituents, (1.68),

$$\mathbf{F} = \sum_a \left[e_a \mathbf{E}(\mathbf{r}_a) + e_a \frac{\mathbf{v}_a}{c} \times \mathbf{B}(\mathbf{r}_a) \right]. \tag{4.2}$$

Since the system is small, all the charges are near the center of mass of the charge distribution, which lies at the position $\mathbf{R}$. (For the purposes of the following expansion we could let $\mathbf{R}$ represent an arbitrary point inside the charge distribution; the use of the center of mass allows us to separate intrinsic properties from those due to the motion of the atom as a whole.) We can then expand the electric and magnetic fields about this reference point,

$$\mathbf{E}(\mathbf{r}_a) = \mathbf{E}(\mathbf{R}) + [(\mathbf{r}_a - \mathbf{R})\cdot\boldsymbol{\nabla}]\mathbf{E}(\mathbf{R}) + \dots, \tag{4.3}$$

and likewise for $\mathbf{B}$, in which the subsequent terms are considered negligible. Here $\boldsymbol{\nabla}$ means the gradient with respect to $\mathbf{R}$. Now, the total force on the atom, (4.2), can be rewritten in terms of this expansion as

$$\begin{aligned}\mathbf{F} &= \left(\sum_a e_a\right)\mathbf{E}(\mathbf{R}) + \sum_a e_a[(\mathbf{r}_a - \mathbf{R})\cdot\boldsymbol{\nabla}]\mathbf{E}(\mathbf{R}) + \left(\sum_a e_a\frac{\mathbf{v}_a}{c}\right)\times\mathbf{B}(\mathbf{R}) \\ &\quad + \sum_a e_a\frac{\mathbf{v}_a}{c}\times[(\mathbf{r}_a - \mathbf{R})\cdot\boldsymbol{\nabla}]\mathbf{B}(\mathbf{R}) + \dots .\end{aligned} \tag{4.4}$$

The first term here is zero because of the neutrality of the system, (4.1). In the second term, we identify the electric dipole moment, $\mathbf{d}$,

$$\mathbf{d} = \sum_a e_a(\mathbf{r}_a - \mathbf{R}) = \sum_a e_a\mathbf{r}_a, \tag{4.5}$$

(which is independent of $\mathbf{R}$), while in the third, we recognize its time derivative,

$$\sum_a e_a\mathbf{v}_a = \frac{d}{dt}\mathbf{d}. \tag{4.6}$$

Momentarily setting aside the fourth term, we find the force on the system to be

$$\mathbf{F} = (\mathbf{d}\cdot\boldsymbol{\nabla})\mathbf{E}(\mathbf{R}) + \frac{1}{c}\left(\frac{d}{dt}\mathbf{d}\right)\times\mathbf{B}(\mathbf{R}) + \dots . \tag{4.7}$$

For the second term here, we can transfer the time derivative,

$$\frac{1}{c}\left(\frac{d}{dt}\mathbf{d}\right)\times\mathbf{B}(\mathbf{R}) = \frac{1}{c}\frac{d}{dt}[\mathbf{d}\times\mathbf{B}(\mathbf{R})] - \frac{1}{c}\mathbf{d}\times\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\boldsymbol{\nabla}\right)\mathbf{B}(\mathbf{R}), \tag{4.8}$$

where $\mathbf{V} = d\mathbf{R}/dt$. Using (1.64) for $\partial\mathbf{B}/\partial t$, and rewriting the resulting double cross product according to

$$\mathbf{d}\times(\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{R})) + (\mathbf{d}\cdot\boldsymbol{\nabla})\mathbf{E}(\mathbf{R}) = \boldsymbol{\nabla}(\mathbf{d}\cdot\mathbf{E}(\mathbf{R})), \tag{4.9}$$

we can present (4.7) as

$$\mathbf{F} = \boldsymbol{\nabla}[\mathbf{d}\cdot\mathbf{E}(\mathbf{R})] - \frac{1}{c}(\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{d}\times\mathbf{B}(\mathbf{R}) + \frac{1}{c}\frac{d}{dt}[\mathbf{d}\times\mathbf{B}(\mathbf{R})] + \dots . \tag{4.10}$$

Recalling that force is the time rate of change of momentum, we see that $(1/c)\mathbf{d}\times\mathbf{B}$ introduces a redefinition of the momentum of the system.

We now return to the fourth term of (4.4), which would seem to correspond to a small effect, since for atomic systems, $v_a/c \ll 1$. A rearrangement of it is

$$\sum_a e_a \frac{\mathbf{v}_a}{c}\times[(\mathbf{r}_a - \mathbf{R})\cdot\boldsymbol{\nabla}]\mathbf{B}(\mathbf{R}) = \sum_a \frac{e_a}{c}\mathbf{V}\times[(\mathbf{r}_a - \mathbf{R})\cdot\boldsymbol{\nabla}]\mathbf{B}(\mathbf{R}) + \sum_a \frac{e_a}{c}(\mathbf{v}_a - \mathbf{V})\times[(\mathbf{r}_a - \mathbf{R})\cdot\boldsymbol{\nabla}]\mathbf{B}(\mathbf{R}), \tag{4.11}$$

where, recalling the definition of the electric dipole moment (4.5), we can express the first term on the right side as

$$\frac{1}{c}\mathbf{V}\times(\mathbf{d}\cdot\boldsymbol{\nabla})\mathbf{B}(\mathbf{R}). \tag{4.12}$$

Combining this contribution with the second term on the right side of (4.10), and using (1.54), we obtain

$$\frac{1}{c}[(\mathbf{d}\cdot\boldsymbol{\nabla})\mathbf{V} - (\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{d}]\times\mathbf{B}(\mathbf{R}) = \frac{1}{c}[(\mathbf{d}\times\mathbf{V})\times\boldsymbol{\nabla}]\times\mathbf{B}(\mathbf{R}) = \boldsymbol{\nabla}\left[\frac{1}{c}(\mathbf{d}\times\mathbf{V})\cdot\mathbf{B}(\mathbf{R})\right]. \tag{4.13}$$

Collecting the various results to this point, we can now rewrite the total force on the atom, (4.4), as

$$\mathbf{F} = \boldsymbol{\nabla}[\mathbf{d}\cdot\mathbf{E}(\mathbf{R})] + \boldsymbol{\nabla}\left[\frac{1}{c}(\mathbf{d}\times\mathbf{V})\cdot\mathbf{B}(\mathbf{R})\right] + \frac{d}{dt}\left[\frac{1}{c}\mathbf{d}\times\mathbf{B}(\mathbf{R})\right] + \mathbf{F}_B, \tag{4.14}$$

where $\mathbf{F}_B$ represents the second term on the right side of (4.11),

$$\mathbf{F}_B = \sum_a \frac{e_a}{c}(\mathbf{v}_a - \mathbf{V})\times[(\mathbf{r}_a - \mathbf{R})\cdot\boldsymbol{\nabla}]\mathbf{B}(\mathbf{R}), \tag{4.15}$$

which can be rearranged as follows:

$$\begin{aligned}\mathbf{F}_B = {} & \frac{d}{dt}\sum_a \frac{e_a}{c}(\mathbf{r}_a - \mathbf{R})\times[(\mathbf{r}_a - \mathbf{R})\cdot\boldsymbol{\nabla}]\mathbf{B}(\mathbf{R}) \\ & - \sum_a \frac{e_a}{c}(\mathbf{r}_a - \mathbf{R})\times[(\mathbf{v}_a - \mathbf{V})\cdot\boldsymbol{\nabla}]\,\mathbf{B}(\mathbf{R}) \\ & - \sum_a \frac{e_a}{c}(\mathbf{r}_a - \mathbf{R})\times[(\mathbf{r}_a - \mathbf{R})\cdot\boldsymbol{\nabla}]\,\frac{d}{dt}\mathbf{B}(\mathbf{R}).\end{aligned} \tag{4.16}$$

We must now recall the restricted nature of this description: The electric and magnetic fields change only slightly over the dimensions of the system. The first of the three terms on the right side of (4.16) is a small correction to what is already present in (4.14) as $\frac{d}{dt}[(1/c)\mathbf{d}\times\mathbf{B}(\mathbf{R})]$, and is therefore to be neglected. Furthermore, the last term of (4.16), which is well approximated by

$$\sum_a e_a(\mathbf{r}_a - \mathbf{R})\times[(\mathbf{r}_a - \mathbf{R})\cdot\boldsymbol{\nabla}]\,\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{R}), \tag{4.17}$$

is of the same order of magnitude as the omitted terms in the expansion (4.4), and is therefore also to be neglected. An average of the initial form of $\mathbf{F}_B$, (4.15), with the single remaining contribution of (4.16), the second line there, now gives

$$\begin{aligned}\mathbf{F}_B &= \frac{1}{2}\sum_a \frac{e_a}{c}(\mathbf{v}_a - \mathbf{V})\times[(\mathbf{r}_a - \mathbf{R})\cdot\boldsymbol{\nabla}]\mathbf{B}(\mathbf{R}) \\ &\quad - \frac{1}{2}\sum_a \frac{e_a}{c}(\mathbf{r}_a - \mathbf{R})\times[(\mathbf{v}_a - \mathbf{V})\cdot\boldsymbol{\nabla}]\,\mathbf{B}(\mathbf{R}) \\ &= \frac{1}{2}\sum_a \frac{e_a}{c}\{[(\mathbf{r}_a - \mathbf{R})\times(\mathbf{v}_a - \mathbf{V})]\times\boldsymbol{\nabla}\}\times\mathbf{B}(\mathbf{R}).\end{aligned} \tag{4.18}$$

What has finally emerged here is the magnetic dipole moment of the system, $\boldsymbol{\mu}$,

$$\boldsymbol{\mu} = \frac{1}{2c}\sum_a e_a(\mathbf{r}_a - \mathbf{R})\times(\mathbf{v}_a - \mathbf{V}), \tag{4.19}$$

so (4.18) is equal to ($\boldsymbol{\mu}$ is constant in space)

$$\mathbf{F}_B = (\boldsymbol{\mu}\times\boldsymbol{\nabla})\times\mathbf{B}(\mathbf{R}) = \boldsymbol{\nabla}[\boldsymbol{\mu}\cdot\mathbf{B}(\mathbf{R})], \tag{4.20}$$

where we have used (1.54). (It is, of course, the similarity of this structure to $\boldsymbol{\nabla}[\mathbf{d}\cdot\mathbf{E}(\mathbf{R})]$ that justifies the identification of $\boldsymbol{\mu}$ as the magnetic analogue of $\mathbf{d}$.) We also recognize that a contribution of this form already appears in the second term on the right side of (4.14), bearing the information that a moving electric dipole also acts as a magnetic dipole. The comparison of the two effects, characterized by $\frac{1}{c}\mathbf{d}\times\mathbf{V}$ and $\boldsymbol{\mu}$, is that of the typical speeds of the relatively heavy atoms, $|\mathbf{V}|$, and of the light electrons, $|\mathbf{v}_a|$, in the interior of atoms,

$$|\mathbf{V}| \ll |\mathbf{v}_a| \ll c. \tag{4.21}$$

Accordingly, we neglect the motional effects of the atoms, and finally write (4.14) as

$$\mathbf{F} = \boldsymbol{\nabla}[\mathbf{d}\cdot\mathbf{E}(\mathbf{R}) + \boldsymbol{\mu}\cdot\mathbf{B}(\mathbf{R})] + \frac{d}{dt}\left(\frac{1}{c}\mathbf{d}\times\mathbf{B}(\mathbf{R})\right). \tag{4.22}$$

In the absence of time variation, what remains is a force associated with the respective potential energies of a given electric dipole in an electric field,

$$-\mathbf{d}\cdot\mathbf{E} \tag{4.23}$$

and of a given magnetic dipole in a magnetic field,

$$-\boldsymbol{\mu}\cdot\mathbf{B}. \tag{4.24}$$

The energy interpretation does more than supply the force components as negative gradients with respect to position coordinates. It also produces torques as negative gradients with respect to angles. Take the example of a magnetic dipole $\boldsymbol{\mu}$ in the presence of a magnetic field $\mathbf{B}$. If θ is the angle between $\boldsymbol{\mu}$ and $\mathbf{B}$, the magnetic potential energy is

$$-|\boldsymbol{\mu}||\mathbf{B}|\cos\theta. \tag{4.25}$$

The implied internal torque, that is, the torque on this individual dipole, and not the moment of the force on the dipole, is then

$$\frac{\partial}{\partial\theta}\left(|\boldsymbol{\mu}||\mathbf{B}|\cos\theta\right) = -|\boldsymbol{\mu}||\mathbf{B}|\sin\theta, \tag{4.26}$$

(the reference point of this torque is at the position of the dipole), which can be represented by a vector perpendicular to the plane formed by $\boldsymbol{\mu}$ and $\mathbf{B}$,

$$\boldsymbol{\tau} = \boldsymbol{\mu}\times\mathbf{B}. \tag{4.27}$$

We shall now derive this vectorial result directly, along with its electric counterpart; for simplicity, additional time derivative terms are omitted. The torque, the moment of the force about the center of the charge distribution at $\mathbf{R}$ is

$$\boldsymbol{\tau} = \sum_a (\mathbf{r}_a - \mathbf{R})\times\left(e_a\mathbf{E}(\mathbf{r}_a) + \frac{1}{c}e_a\mathbf{v}_a\times\mathbf{B}(\mathbf{r}_a)\right). \tag{4.28}$$

The part proportional to the electric field is, when we neglect the variation of $\mathbf{E}$ over the system, the electric torque

$$\boldsymbol{\tau}_E = \mathbf{d}\times\mathbf{E}(\mathbf{R}), \tag{4.29}$$

as expected in analogy with (4.27). In deriving the magnetic torque, we first make the unimportant change, $\mathbf{v}_a \to \mathbf{v}_a - \mathbf{V}$, using (4.21), and then transfer the time derivative to get

$$\begin{aligned}
\boldsymbol{\tau}_B &= \sum_a (\mathbf{r}_a - \mathbf{R})\times\left(\frac{1}{c}e_a(\mathbf{v}_a - \mathbf{V})\times\mathbf{B}(\mathbf{R})\right) \\
&\to -\sum_a (\mathbf{v}_a - \mathbf{V})\times\left[\frac{1}{c}e_a(\mathbf{r}_a - \mathbf{R})\times\mathbf{B}(\mathbf{R})\right] \\
&\to \frac{1}{2}\sum_a \frac{e_a}{c}\left\{(\mathbf{r}_a - \mathbf{R})\times[(\mathbf{v}_a - \mathbf{V})\times\mathbf{B}(\mathbf{R})] - (\mathbf{v}_a - \mathbf{V})\times[(\mathbf{r}_a - \mathbf{R})\times\mathbf{B}(\mathbf{R})]\right\} \\
&= \boldsymbol{\mu}\times\mathbf{B}(\mathbf{R}),
\end{aligned} \tag{4.30}$$

where in the second line we have omitted the $-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{E}$ contribution as negligible in comparison with $\boldsymbol{\tau}_E$. [See Problem 4.2 for a justification of (4.30).] In the third line, we averaged the two preceding forms, and then used the first identity in Problem 1.1. Putting all this together, we find the torque on the system is given by

$$\boldsymbol{\tau} = \mathbf{d}\times\mathbf{E} + \boldsymbol{\mu}\times\mathbf{B}, \tag{4.31}$$

so that, as with the force, the result can be expressed in terms of the electric and magnetic dipole moments, $\mathbf{d}$ and $\boldsymbol{\mu}$.

4.2 Force on a Macroscopic Body

To this point, we have considered the response of a small system, an atom, for example, to external electric and magnetic fields, which vary smoothly over the system. Macroscopic materials are made up of large numbers of atoms. What is the total force on such a piece of material? We must sum up all the forces on the individual atoms. To the extent that the forces on the atoms vary but slightly from one atom to another, the summation can be replaced by a volume integration, weighted by the atomic density, $n(\mathbf{r})$, the number of atoms per unit volume at the macroscopic point $\mathbf{r}$:

$$\begin{aligned}\mathbf{F} \,=\, \int (d\mathbf{r})\, n(\mathbf{r}) \bigg[&\mathbf{d}\times(\boldsymbol{\nabla}\times\mathbf{E}) + (\mathbf{d}\cdot\boldsymbol{\nabla})\mathbf{E} + \boldsymbol{\mu}\times(\boldsymbol{\nabla}\times\mathbf{B}) + (\boldsymbol{\mu}\cdot\boldsymbol{\nabla})\mathbf{B} \\ &+ \frac{d}{dt}\left(\frac{1}{c}\mathbf{d}\times\mathbf{B}\right)\bigg]. \end{aligned} \tag{4.32}$$

Notice that we have rewritten (4.22) with the aid of the identities

$$\boldsymbol{\nabla}(\mathbf{d}\cdot\mathbf{E}) \,=\, \mathbf{d}\times(\boldsymbol{\nabla}\times\mathbf{E}) + (\mathbf{d}\cdot\boldsymbol{\nabla})\mathbf{E}, \tag{4.33}$$

$$\boldsymbol{\nabla}(\boldsymbol{\mu}\cdot\mathbf{B}) \,=\, \boldsymbol{\mu}\times(\boldsymbol{\nabla}\times\mathbf{B}) + (\boldsymbol{\mu}\cdot\boldsymbol{\nabla})\mathbf{B}. \tag{4.34}$$

First a word about $\mathbf{d}$ and $\boldsymbol{\mu}$ in these expressions. In the single atom formula (4.22), the derivatives act only on $\mathbf{E}$ and $\mathbf{B}$, which is reflected in (4.32). For a many-atom system, the dipole moments could well vary from one location to another and so have macroscopic spatial dependence. Accordingly, $\mathbf{d}(\mathbf{r})$ and $\boldsymbol{\mu}(\mathbf{r})$ are the average dipole moments at the point $\mathbf{r}$. We now define the electric polarization, $\mathbf{P}$, and the magnetization, $\mathbf{M}$, by

$$\mathbf{P}(\mathbf{r},t) = n(\mathbf{r})\mathbf{d}(\mathbf{r},t), \tag{4.35}$$

and

$$\mathbf{M}(\mathbf{r},t) = n(\mathbf{r})\boldsymbol{\mu}(\mathbf{r},t), \tag{4.36}$$

respectively. The resulting macroscopic form of the total force at time t is

$$\begin{aligned}\mathbf{F}(t) \,=\, \int (d\mathbf{r}) \bigg[&\mathbf{P}(\mathbf{r},t)\times[\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r},t)] + [\mathbf{P}(\mathbf{r},t)\cdot\boldsymbol{\nabla}]\mathbf{E}(\mathbf{r},t) \\ &+ \mathbf{M}(\mathbf{r},t)\times[\boldsymbol{\nabla}\times\mathbf{B}(\mathbf{r},t)] + [\mathbf{M}(\mathbf{r},t)\cdot\boldsymbol{\nabla}]\mathbf{B}(\mathbf{r},t) \\ &+ \frac{\partial}{\partial t}\left(\frac{1}{c}\mathbf{P}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t)\right)\bigg]. \end{aligned} \tag{4.37}$$

(Here, the distinction between $\frac{d}{dt}\mathbf{B}$ and $\frac{\partial}{\partial t}\mathbf{B}$ has been dropped, because the difference is of order of the small atomic velocity $\mathbf{V}$, which is averaged to zero in any case.)

We proceed to simplify this in various ways. First, we use one of Maxwell's equations to obtain

$$\mathbf{P}\times(\boldsymbol{\nabla}\times\mathbf{E}) + \frac{\partial}{\partial t}\left(\frac{1}{c}\mathbf{P}\times\mathbf{B}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}\mathbf{P}\right)\times\mathbf{B}, \tag{4.38}$$

and then we use the identity

$$\boldsymbol{\nabla}(\mathbf{M}\cdot\mathbf{B}) = \mathbf{M}\times(\boldsymbol{\nabla}\times\mathbf{B}) + (\mathbf{M}\cdot\boldsymbol{\nabla})\mathbf{B} + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{M}) + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{M}, \tag{4.39}$$

which is a generalization of (4.34). All subsequent steps involve the statement that the integral is extended over a volume that includes the whole body, so that, on the bounding surface of that volume, $n(\mathbf{r}) = 0$. This means that in performing partial integrations through the use of the divergence theorem, the surface integrals vanish. In effect, then,

$$(\mathbf{P}\cdot\boldsymbol{\nabla})\mathbf{E} \to -(\boldsymbol{\nabla}\cdot\mathbf{P})\mathbf{E}, \tag{4.40}$$

and similarly, using $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, (4.39) yields

$$\mathbf{M}\times(\boldsymbol{\nabla}\times\mathbf{B}) + (\mathbf{M}\cdot\boldsymbol{\nabla})\mathbf{B} \to (\boldsymbol{\nabla}\times\mathbf{M})\times\mathbf{B}. \tag{4.41}$$

The immediate result is

$$\mathbf{F} = \int (d\mathbf{r}) \left[-(\boldsymbol{\nabla}\cdot\mathbf{P})\mathbf{E} + \frac{1}{c}\left(\frac{\partial}{\partial t}\mathbf{P}\right)\times\mathbf{B} + (\boldsymbol{\nabla}\times\mathbf{M})\times\mathbf{B}\right]. \tag{4.42}$$

The comparison of this with the microscopic description of the force on charge and current densities, (3.8) for zero magnetic charge, suggests the definition of an effective charge density, ρ_{eff}, and an effective current density, $\mathbf{j}_{\text{eff}}$, as

$$\rho_{\text{eff}}(\mathbf{r},t) = -\boldsymbol{\nabla}\cdot\mathbf{P}(\mathbf{r},t), \tag{4.43}$$

$$\mathbf{j}_{\text{eff}}(\mathbf{r},t) = \frac{\partial}{\partial t}\mathbf{P}(\mathbf{r},t) + c\boldsymbol{\nabla}\times\mathbf{M}(\mathbf{r},t). \tag{4.44}$$

Notice that these effective densities satisfy the equation of charge conservation,

$$\frac{\partial}{\partial t}\rho_{\text{eff}} + \boldsymbol{\nabla}\cdot\mathbf{j}_{\text{eff}} = 0. \tag{4.45}$$

It is left to the reader to verify (Problem 4.3) that the total torque, $\boldsymbol{\tau}$, on the body, the sum over all atoms of the external torques:

$$\boldsymbol{\tau}_{\text{ext}} = \int (d\mathbf{r})\, n\mathbf{r}\times\left[\mathbf{d}\times(\boldsymbol{\nabla}\times\mathbf{E}) + (\mathbf{d}\cdot\boldsymbol{\nabla})\mathbf{E} + \boldsymbol{\mu}\times(\boldsymbol{\nabla}\times\mathbf{B}) + (\boldsymbol{\mu}\cdot\boldsymbol{\nabla})\mathbf{B} + \frac{1}{c}\frac{d}{dt}(\mathbf{d}\times\mathbf{B})\right], \tag{4.46}$$

and of the internal torques:

$$\boldsymbol{\tau}_{\rm int} = \int (d\mathbf{r})\, n(\mathbf{d}\times\mathbf{E} + \boldsymbol{\mu}\times\mathbf{B}), \tag{4.47}$$

is properly reproduced as the integrated moment of the effective force density,

$$\boldsymbol{\tau} = \int (d\mathbf{r})\, \mathbf{r}\times \left[\rho_{\rm eff}\mathbf{E} + \frac{1}{c}\mathbf{j}_{\rm eff}\times\mathbf{B}\right]. \tag{4.48}$$

4.3 Macroscopic Electrodynamics

Now we construct a phenomenological macroscopic electrodynamics. And what is that? Nothing more than the form in which electrodynamics first arose, in the pre-atomic period, when only the properties of bulk matter were involved. But the challenge here is to derive the phenomenological theory from the microscopic Maxwell-Lorentz description. Both theories will employ concepts that are abstracted from the kinds of measurements that are appropriate to their level of description. The microscopic regime is characterized by rapid space-time variations unlike the macroscopic one, which is characterized by scales large compared to those of atoms. Laboratory instruments, being large, directly measure average quantities. Macroscopic fields are thus defined in terms of averages over space and time intervals, V and T, large on the atomic scale but small compared to typical macroscopic intervals. We adopt the convention that lower-case letters, like $f(\mathbf{r},t)$, represent microscopic quantities while capital letters, like $F(\mathbf{r},t)$, represent the corresponding macroscopic quantities. The connection between the two is

$$F(\mathbf{r},t) = \frac{1}{T}\int_T dt'\,\frac{1}{V}\int_V (d\mathbf{r}')\, f(\mathbf{r}+\mathbf{r}', t+t') = \overline{f(\mathbf{r},t)}. \tag{4.49}$$

This is a linear relation, in the sense that

$$\overline{f_1 + f_2} = \overline{f_1} + \overline{f_2}, \quad \overline{\lambda f} = \lambda \overline{f}, \tag{4.50}$$

where λ is a constant. From this follows the connection between derivatives of microscopic and macroscopic quantities, that is, that the averaged derivative of a function is the derivative of the average:

$$\begin{aligned} \frac{\partial}{\partial t}\overline{f(\mathbf{r},t)} &= \overline{\frac{\partial}{\partial t} f(\mathbf{r},t)}, \\ \boldsymbol{\nabla}\overline{f(\mathbf{r},t)} &= \overline{\boldsymbol{\nabla} f(\mathbf{r},t)}. \end{aligned} \tag{4.51}$$

The microscopic charge distribution is composed of two parts. That which is confined to atoms is called bound charge. When the remaining, "free," microscopic charge distributions are appropriately averaged, we obtain the macroscopic densities

$$\rho = \overline{\rho_{\rm free}}, \qquad \mathbf{J} = \overline{\mathbf{j}_{\rm free}}. \tag{4.52}$$

electric field	magnetic field	charge density	current density
$\mathbf{e}$	$\mathbf{b}$	$\rho_{\text{free}} + \rho_{\text{bound}}$	$\mathbf{j}_{\text{free}} + \mathbf{j}_{\text{bound}}$
$\mathbf{E}$	$\mathbf{B}$	$\rho - \boldsymbol{\nabla}\cdot\mathbf{P}$	$\mathbf{J} + \frac{\partial}{\partial t}\mathbf{P} + c\boldsymbol{\nabla}\times\mathbf{M}$

Table 4.1: Connection between microscopic and macroscopic quantities

What is the macroscopic role of the bound charge distributions? It must be related to the effective charge and current densities given in terms of the polarization and the magnetization by (4.43) and (4.44),

$$\begin{aligned} \rho_{\text{eff}} &= -\boldsymbol{\nabla}\cdot\mathbf{P}, && (4.53)\\ \mathbf{j}_{\text{eff}} &= \frac{\partial}{\partial t}\mathbf{P} + c\boldsymbol{\nabla}\times\mathbf{M}. && (4.54) \end{aligned}$$

As we have seen in the preceding section, these densities are examples of macroscopically measured quantities, disclosed by slowly varying electric and magnetic fields. The physical measurements necessary for the definitions of ρ_{eff} and $\mathbf{j}_{\text{eff}}$, since they employ slowly varying fields, should correspond to the mathematical process of averaging involved in the definitions of $\overline{\rho_{\text{bound}}}$ and $\overline{\mathbf{j}_{\text{bound}}}$, so we have the identifications

$$\begin{aligned} \overline{\rho_{\text{bound}}} &= \rho_{\text{eff}}\\ \overline{\mathbf{j}_{\text{bound}}} &= \mathbf{j}_{\text{eff}}. && (4.55) \end{aligned}$$

In view of (4.45), these two forms of macroscopic charge are separately conserved.

The correspondence between microscopic and macroscopic quantities is given by Table 4.1: The microscopic Maxwell equations now read

$$\begin{aligned} \boldsymbol{\nabla}\times\mathbf{b} &= \frac{1}{c}\frac{\partial}{\partial t}\mathbf{e} + \frac{4\pi}{c}(\mathbf{j}_{\text{free}} + \mathbf{j}_{\text{bound}}), && \boldsymbol{\nabla}\cdot\mathbf{e} = 4\pi(\rho_{\text{free}} + \rho_{\text{bound}}),\\ -\boldsymbol{\nabla}\times\mathbf{e} &= \frac{1}{c}\frac{\partial}{\partial t}\mathbf{b}, && \boldsymbol{\nabla}\cdot\mathbf{b} = 0. \qquad (4.56) \end{aligned}$$

These are averaged to yield the macroscopic equations,

$$\begin{aligned} \boldsymbol{\nabla}\times\mathbf{B} &= \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} + \frac{4\pi}{c}\left(\mathbf{J} + \frac{\partial}{\partial t}\mathbf{P} + c\boldsymbol{\nabla}\times\mathbf{M}\right), && \boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi(\rho - \boldsymbol{\nabla}\cdot\mathbf{P}),\\ -\boldsymbol{\nabla}\times\mathbf{E} &= \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}, && \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \qquad (4.57) \end{aligned}$$

which can be cast into the form of the microscopic equations if we define the displacement, $\mathbf{D}$,

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \qquad (4.58)$$

and the magnetic field, $\mathbf{H}$,

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} \tag{4.59}$$

(recall that $\mathbf{B}$ is properly called the magnetic induction). The final form of the historical, macroscopic Maxwell equations is

$$\begin{aligned} \boldsymbol{\nabla}\times\mathbf{H} &= \frac{1}{c}\frac{\partial}{\partial t}\mathbf{D} + \frac{4\pi}{c}\mathbf{J}, \qquad && \boldsymbol{\nabla}\cdot\mathbf{D} = 4\pi\rho, \\ -\boldsymbol{\nabla}\times\mathbf{E} &= \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}, && \boldsymbol{\nabla}\cdot\mathbf{B} = 0. \end{aligned} \tag{4.60}$$

Note that the macroscopic charge is conserved,

$$\boldsymbol{\nabla}\cdot\mathbf{J} + \frac{\partial}{\partial t}\rho = 0, \tag{4.61}$$

which follows from the first pair of equations. As microscopically smooth distributions, the density and flux of free charge will serve to measure the macroscopic fields $\mathbf{E}$ and $\mathbf{B}$. That is exhibited in the expression for the force on a macroscopic charge distribution,

$$\mathbf{F} = \int (d\mathbf{r}) \left(\rho\mathbf{E} + \frac{1}{c}\mathbf{J}\times\mathbf{B}\right). \tag{4.62}$$

[If bound charge is present, there is an additional contribution to the force coming from (4.42).]

For a complete description of the system, we require further relations between $\mathbf{D}$, $\mathbf{E}$, $\mathbf{P}$, and $\mathbf{J}$, expressing how material bodies respond to electric fields. Similar remarks hold for $\mathbf{H}$, $\mathbf{B}$, and $\mathbf{M}$. These constitutive relations depend on the characteristics of the particular material under consideration. Simple classical models—which are not qualitatively misleading—will be considered in the following two chapters.

4.4 Problems for Chapter 4

1. Find the total charge and the dipole moment of the charge density

$$\rho(\mathbf{r}) = -\mathbf{d}\cdot\boldsymbol{\nabla}\delta(\mathbf{r}).$$

2. Justify the approximation leading to the final form of $\boldsymbol{\tau}$ in (4.30). In particular, show that the total time derivative omitted in going from the first to the second line of (4.30) leads to

$$\frac{d}{dt}\sum_a \mathbf{r}_a\times\mathbf{p}_a = \boldsymbol{\tau},$$

where the "canonical momentum" $\mathbf{p}_a$ is defined by

$$\mathbf{p}_a = m_a\mathbf{v}_a + \frac{e_a}{c}\mathbf{A}(\mathbf{r}_a),$$

where the vector potential $\mathbf{A}$ for a constant magnetic field $\mathbf{B}$ is

$$\mathbf{A} = -\frac{1}{2}\mathbf{r}\times\mathbf{B}, \quad \boldsymbol{\nabla}\times\mathbf{A} = \mathbf{B}.$$

3. By summing the torque on an individual charge,

$$\boldsymbol{\tau}_a = \mathbf{r}_a\times\left(e_a\mathbf{E}(\mathbf{r}_a) + e_a\frac{\mathbf{v}_a}{c}\times\mathbf{B}(\mathbf{r}_a)\right),$$

first, over the charges in an individual atom, and thereby obtaining expressions in terms of $\mathbf{d}_a$ and $\boldsymbol{\mu}_a$, the dipole moments of the atom, and then over the atoms making up a macroscopic body, obtain the result that

$$\boldsymbol{\tau} = \boldsymbol{\tau}_{\text{ext}} + \boldsymbol{\tau}_{\text{int}},$$

where the external and internal torques are given by (4.46) and (4.47), respectively. Then, verify that the torque acting on a macroscopic object in electric and magnetic fields is given in terms of ρ_{eff} and $\mathbf{j}_{\text{eff}}$ according to (4.48).