Chapter 3

Conservation Laws

In order to check the physical consistency of the above set of equations governing Maxwell-Lorentz electrodynamics [(2.10) and (2.12) or (1.65) and (1.68)], we examine the action of, and reaction on, the sources of the electromagnetic fields. To be precise, we ask whether there is a correct balance in the exchange of energy, momentum, and angular momentum between the charged particles and the electromagnetic fields. As we shall see, the Maxwell-Lorentz system as it stands implies the conservation of these mechanical properties, no matter how rapidly the charges are moving.

3.1 Conservation of Energy

We start with a consideration of the rate at which work is done on the particles, that is, the rate of energy transfer, or the power absorbed by the particles. For one particle, we know that the rate at which work is done on it is

$$\mathbf{F}\cdot\mathbf{v} = e\mathbf{v}\cdot\mathbf{E} + g\mathbf{v}\cdot\mathbf{B} = \int (d\mathbf{r})\,(\mathbf{j}_e\cdot\mathbf{E} + \mathbf{j}_m\cdot\mathbf{B}), \tag{3.1}$$

where we have used the Lorentz force law, (2.12), and the expressions for the currents, (1.44) and (2.7), for a point particle. We interpret this equation as meaning, even for general current distributions, that $\mathbf{j}_e\cdot\mathbf{E}+\mathbf{j}_m\cdot\mathbf{B}$ is the rate of energy transfer from the field to the particles, per unit volume. Then through elimination of the currents by use of Maxwell's equations, (2.10), this rate can be rewritten as

$$\begin{aligned}\mathbf{j}_e\cdot\mathbf{E} + \mathbf{j}_m\cdot\mathbf{B} &= \frac{c}{4\pi}\left(\boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}\right)\cdot\mathbf{E} + \frac{c}{4\pi}\left(-\boldsymbol{\nabla}\times\mathbf{E} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}\right)\cdot\mathbf{B}\\ &= -\frac{\partial}{\partial t}\left(\frac{E^2+B^2}{8\pi}\right) - \boldsymbol{\nabla}\cdot\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{B}\right). \end{aligned}\tag{3.2}$$

The general form of any local conservation law, (1.45) or (1.46), suggests the following interpretations:

1. In the absence of charges ($\mathbf{j}_e = \mathbf{j}_m = \mathbf{0}$), this is the local energy conservation law
$$\frac{\partial}{\partial t}\frac{E^2+B^2}{8\pi} + \boldsymbol{\nabla}\cdot\frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = 0. \tag{3.3}$$
We label the two objects appearing here as
$$\text{energy density} = U = \frac{E^2+B^2}{8\pi}, \tag{3.4}$$
$$\text{energy flux vector} = \mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}. \tag{3.5}$$
[The latter is usually called the Poynting vector, after John Henry Poynting (1852–1914).]

2. In the presence of charges, the relation (3.2) is
$$\frac{\partial}{\partial t}U + \boldsymbol{\nabla}\cdot\mathbf{S} + \mathbf{j}_e\cdot\mathbf{E} + \mathbf{j}_m\cdot\mathbf{B} = 0, \tag{3.6}$$
which, if we integrate over an arbitrary volume V, bounded by a surface S, becomes
$$\frac{d}{dt}\int_V (d\mathbf{r})\,U + \oint_S d\mathbf{S}\cdot\mathbf{S} + \int_V (d\mathbf{r})\,(\mathbf{j}_e\cdot\mathbf{E} + \mathbf{j}_m\cdot\mathbf{B}) = 0. \tag{3.7}$$
The three terms here are identified, respectively, as the rate of change of the electromagnetic field energy within the volume, the rate of flow of electromagnetic energy out of the volume, and the rate of transfer of electromagnetic energy to the charged particles. Thus, (3.6) gives a complete description of energy conservation.

3.2 Conservation of Momentum

Next we consider the force on a particle, (2.12), as the rate of change of momentum,
$$\begin{aligned}\mathbf{F} &= e\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right) + g\left(\mathbf{B} - \frac{\mathbf{v}}{c}\times\mathbf{E}\right)\\ &= \int (d\mathbf{r})\,\left(\rho_e\mathbf{E} + \frac{1}{c}\mathbf{j}_e\times\mathbf{B} + \rho_m\mathbf{B} - \frac{1}{c}\mathbf{j}_m\times\mathbf{E}\right)\\ &\equiv \int (d\mathbf{r})\,\mathbf{f},\end{aligned} \tag{3.8}$$
where $\mathbf{f}$ is the force density. Removing reference to the (generalized) charge and current densities by use of Maxwell's equations, (2.10), we rewrite the force density $\mathbf{f}$ as
$$\mathbf{f} = \frac{1}{4\pi}[\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E}) + \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{B})]$$

$$+\frac{1}{4\pi}\left[\left(-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}+\boldsymbol{\nabla}\times\mathbf{B}\right)\times\mathbf{B}+\mathbf{E}\times\left(-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}-\boldsymbol{\nabla}\times\mathbf{E}\right)\right] \tag{3.9}$$

$$= -\frac{\partial}{\partial t}\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}+\frac{1}{4\pi}[-\mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E})+\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E})-\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B})+\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{B})].$$

The quadratic structure in $\mathbf{E}$ occurring here is

$$\begin{aligned}-\mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E})+\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E}) &= -\boldsymbol{\nabla}\frac{E^2}{2}+(\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{E}+\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E})\\ &= \boldsymbol{\nabla}\cdot\left(-\mathbf{1}\frac{E^2}{2}+\mathbf{E}\mathbf{E}\right),\end{aligned} \tag{3.10}$$

which introduces dyadic notation, including the unit dyadic $\mathbf{1}$, with components

$$\mathbf{1}_{kl}=\delta_{kl}=\begin{cases}1, & k=l,\\ 0, & k\neq l,\end{cases} \tag{3.11}$$

where δ_{kl} is the Kronecker δ symbol. (See Problem 3.1.) The analogous result holds for $\mathbf{B}$. Accordingly, the force density is

$$\mathbf{f}=-\frac{\partial}{\partial t}\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}-\boldsymbol{\nabla}\cdot\left(\mathbf{1}\frac{E^2+B^2}{8\pi}-\frac{\mathbf{E}\mathbf{E}+\mathbf{B}\mathbf{B}}{4\pi}\right). \tag{3.12}$$

We interpret this equation physically by identifying

$$\text{momentum density}=\mathbf{G}=\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}, \tag{3.13}$$

and

$$\text{momentum flux (stress tensor)}=\mathbf{T}=\mathbf{1}\frac{E^2+B^2}{8\pi}-\frac{\mathbf{E}\mathbf{E}+\mathbf{B}\mathbf{B}}{4\pi}. \tag{3.14}$$

When $\mathbf{f}=\mathbf{0}$, we obtain the local statement of the conservation of momentum of the electromagnetic field. A full account of momentum balance is contained in

$$\frac{\partial}{\partial t}\mathbf{G}+\boldsymbol{\nabla}\cdot\mathbf{T}+\mathbf{f}=\mathbf{0}. \tag{3.15}$$

The volume integral of this equation for electromagnetic momentum is interpreted analogously to the energy result, (3.7).

The components of the stress tensor are given by

$$T_{kl}=\delta_{kl}U-\frac{E_kE_l+B_kB_l}{4\pi}. \tag{3.16}$$

Notice that the stress tensor is symmetrical, $T_{kl}=T_{lk}$, which, as we shall see in the next section, is required in order to obtain a local conservation law for angular momentum. The trace of $\mathbf{T}$, the sum of the diagonal elements T_{kk}, is simply the energy density, (3.4),

$$\operatorname{Tr}T=\sum_k T_{kk}=U. \tag{3.17}$$

We also note that the Poynting vector, (3.5), is proportional to the momentum density,

$$\mathbf{S} = c^2\mathbf{G}, \tag{3.18}$$

which has the structure of

$$\text{energy density} \times \text{velocity} = c^2(\text{mass density} \times \text{velocity}). \tag{3.19}$$

This is the first indication of the relativistic connection between energy and mass, $E = mc^2$.

3.3 Conservation of Angular Momentum. Virial Theorem

Having discussed momentum, we now turn to angular momentum. We will use tensor notation to write (3.15) in component form,

$$\frac{\partial}{\partial t}G_k + \nabla_l T_{lk} + f_k = 0, \tag{3.20}$$

where we have also used the summation convention: Whenever an index is repeated, a sum over all values of that index is assumed,

$$a_i b_i \equiv \sum_{i=1}^{3} a_i b_i = \mathbf{a} \cdot \mathbf{b}. \tag{3.21}$$

The rate of change of angular momentum is the torque $\boldsymbol{\tau}$, which, for one particle, is

$$\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F} = \int (d\mathbf{r})\,\mathbf{r}\times\mathbf{f}, \tag{3.22}$$

where the volume-integrated form is no longer restricted to a single particle. The torque density, the moment of the force density, can be written in component form as

$$(\mathbf{r}\times\mathbf{f})_i = \epsilon_{ijk} x_j f_k, \tag{3.23}$$

where we have introduced the totally antisymmetric (Levi-Civita) symbol ϵ_{ijk}, which changes sign under any interchange of two indices,

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{kji} = -\epsilon_{ikj} = +\epsilon_{kij} = +\epsilon_{jki}, \tag{3.24}$$

and is normalized by $\epsilon_{123} = 1$. In particular, then, it vanishes if any two indices are equal, $\epsilon_{112} = 0$, for example. The torque density may be obtained by first taking the moment of the force density equation (3.20),

$$\frac{\partial}{\partial t}x_j G_k + \nabla_l(x_j T_{lk}) - T_{jk} + x_j f_k = 0, \tag{3.25}$$

where we have noted that

$$\nabla_l x_j = \delta_{lj}. \tag{3.26}$$

When we now multiply (3.25) with ϵ_{ijk} and sum over repeated indices, we find that the terms involving spatial derivatives can be written as a divergence:

$$\frac{\partial}{\partial t}(\epsilon_{ijk}x_jG_k) + \nabla_l(\epsilon_{ijk}x_jT_{lk}) + \epsilon_{ijk}x_jf_k = 0. \tag{3.27}$$

This final step is justified *only* because T_{kl} is symmetrical (thus this symmetry is required for the existence of a local conservation law of angular momentum). We therefore identify the following electromagnetic angular momentum quantities:

$$\text{angular momentum density} = \boldsymbol{\mathcal{J}} = \mathbf{r}\times\mathbf{G} \tag{3.28}$$

$$\text{angular momentum flux tensor} = \boldsymbol{\mathcal{K}}, \quad \mathcal{K}_{ij} = \epsilon_{jkl}x_kT_{il}. \tag{3.29}$$

The interpretation of (3.27) as a local account of angular momentum conservation for fields and particles proceeds as before. (See Problem 3.5.)

Another important application of (3.25) results if we set $j = k$ and sum. With the aid of (3.17) this gives

$$\frac{\partial}{\partial t}(\mathbf{r}\cdot\mathbf{G}) + \boldsymbol{\nabla}\cdot(\mathbf{T}\cdot\mathbf{r}) - U + \mathbf{r}\cdot\mathbf{f} = 0, \tag{3.30}$$

which we call the electromagnetic virial theorem, in analogy with the mechanical virial theorem of Rudolf Clausius (1822–1888). (See Chapter 8.)

3.4 Conservation Laws and the Speed of Light

In this section, we restrict our attention to electromagnetic fields in domains free of charged particles, specifically, moving, finite regions occupied by electromagnetic fields, which we will refer to as electromagnetic pulses. The total electromagnetic energy of such a pulse is constant in time:

$$\frac{d}{dt}E = \int_{\text{pulse}}(d\mathbf{r})\,\frac{\partial}{\partial t}U = -\int_{\text{pulse}}(d\mathbf{r})\,\boldsymbol{\nabla}\cdot\mathbf{S} = 0, \tag{3.31}$$

inasmuch as the resulting surface integral, conducted over an enclosing surface on which all fields vanish, equals zero. Similar considerations apply to the total electromagnetic linear and angular momentum,

$$\frac{d}{dt}\mathbf{P} = \int_{\text{pulse}}(d\mathbf{r})\,\frac{\partial}{\partial t}\mathbf{G} = -\int_{\text{pulse}}(d\mathbf{r})\,\boldsymbol{\nabla}\cdot\mathbf{T} = \mathbf{0}, \tag{3.32}$$

$$\frac{d}{dt}\mathbf{J} = \int_{\text{pulse}}(d\mathbf{r})\,\frac{\partial}{\partial t}(\mathbf{r}\times\mathbf{G}) = -\int_{\text{pulse}}(d\mathbf{r})\,\boldsymbol{\nabla}\cdot(-\mathbf{T}\times\mathbf{r}) = \mathbf{0}. \tag{3.33}$$

With an eye toward relativity, we consider the space and time moments of (3.6) and (3.15), respectively, combined as a single vector statement:

$$0 = x_k\left(\frac{\partial}{\partial t}U + \boldsymbol{\nabla}\cdot\mathbf{S}\right) - c^2t\left(\frac{\partial}{\partial t}G_k + \nabla_lT_{lk}\right), \tag{3.34}$$

outside the charge and current distributions. Exploiting the connection between **S** and **G** [(3.18)], we can rewrite (3.34) as a local conservation law, much as the equality of T_{jk} and T_{kj} lead to the conservation of angular momentum:

$$\frac{\partial}{\partial t}(\mathbf{r}U - c^2 t\mathbf{G}) + \boldsymbol{\nabla}\cdot(\mathbf{S}\mathbf{r} - c^2 t\mathbf{T}) = \mathbf{0}. \tag{3.35}$$

When (3.35) is integrated over a volume enclosing the electromagnetic pulse, the surface term does not contribute, and we find

$$\frac{d}{dt}\int_{\text{pulse}}(d\mathbf{r})\,(\mathbf{r}U - c^2 t\mathbf{G}) = \mathbf{0}. \tag{3.36}$$

The volume integral of the momentum density is the total momentum **P**,

$$\int_{\text{pulse}}(d\mathbf{r})\,\mathbf{G} = \mathbf{P}, \tag{3.37}$$

which as noted in (3.32) is constant in time. Consequently, we can rewrite (3.36) as

$$\frac{d}{dt}\int_{\text{pulse}}(d\mathbf{r})\,\mathbf{r}U = c^2\mathbf{P}, \tag{3.38}$$

where the integral here provides an energy weighting of the position vector, at each instant of time,

$$\int_{\text{pulse}}(d\mathbf{r})\,\mathbf{r}U(\mathbf{r},t) = E\,\langle\mathbf{r}\rangle_E(t), \tag{3.39}$$

where, as in (3.31), the energy E is

$$E = \int_{\text{pulse}}(d\mathbf{r})\,U. \tag{3.40}$$

Thus the motion of this energy-centroid vector is governed by

$$\frac{E}{c^2}\frac{d}{dt}\langle\mathbf{r}\rangle_E(t) = \mathbf{P}, \tag{3.41}$$

which is to say that the center of energy, $\langle\mathbf{r}\rangle_E(t)$, moves with constant velocity,

$$\frac{d}{dt}\langle\mathbf{r}\rangle_E(t) = \mathbf{v}_E, \tag{3.42}$$

the total momentum being that velocity multiplied by a mass,

$$m = E/c^2. \tag{3.43}$$

The application of the virial theorem, (3.30), to an electromagnetic pulse supplies another velocity. We infer that

$$\frac{d}{dt}\int_{\text{pulse}}(d\mathbf{r})\,\mathbf{r}\cdot\mathbf{G} = E. \tag{3.44}$$

By introducing a momentum weighting for the position vector,

$$\int_{\text{pulse}} (d\mathbf{r})\, \mathbf{r} \cdot \mathbf{G}(\mathbf{r}, t) = \langle \mathbf{r} \rangle_P(t) \cdot \mathbf{P}, \tag{3.45}$$

we deduce that the center of momentum moves with velocity

$$\frac{d}{dt} \langle \mathbf{r} \rangle_P(t) = \mathbf{v}_P, \tag{3.46}$$

which is constant in the direction of the momentum,

$$\mathbf{v}_P \cdot \mathbf{P} = E. \tag{3.47}$$

We combine (3.47) with (3.41) to yield

$$\mathbf{v}_P \cdot \mathbf{v}_E = c^2. \tag{3.48}$$

If the flow of energy and momentum takes place in a single direction, it would be reasonable to expect that these mechanical properties are being transported with a common velocity,

$$\mathbf{v}_E = \mathbf{v}_P = \mathbf{v}, \tag{3.49}$$

which then has a definite magnitude,

$$\mathbf{v} \cdot \mathbf{v} = c^2, \quad v = c, \tag{3.50}$$

which supplies the physical identification of c as the speed of light. Of course, this identification was an input to our inference of Maxwell's equations. We here recover it from a consideration of energy and momentum, thus indicating the consistency of Maxwell's equations. The relation between the momentum and the energy of this electromagnetic pulse is then

$$E = \mathbf{v} \cdot \mathbf{P}, \quad \mathbf{P} = \frac{E}{c^2} \mathbf{v}, \tag{3.51}$$

so we learn that

$$E = Pc, \quad \mathbf{v} = c \frac{\mathbf{P}}{P}, \tag{3.52}$$

which results express the mechanical properties of a localized electromagnetic pulse carrying both energy and momentum at the speed of light, in the direction of the momentum.

There is another, somewhat more direct, mechanical proof that electromagnetic pulses propagate at speed c. When no charges or currents are present, the local equation of energy conservation, (3.3), implies

$$(r^2 - c^2 t^2) \left[\frac{\partial}{\partial t} U + \boldsymbol{\nabla} \cdot \mathbf{S} \right] = 0, \tag{3.53}$$

which can be rewritten, using (3.18), as

$$\frac{\partial}{\partial t} [(r^2 - c^2 t^2) U] + \boldsymbol{\nabla} \cdot [(r^2 - c^2 t^2) \mathbf{S}] + 2c^2 [tU - \mathbf{r} \cdot \mathbf{G}] = 0. \tag{3.54}$$

Integrating this over all space and using the idea of energy and momentum weighting to define averages, as before, we obtain

$$\frac{d}{dt}\left[\langle r^2\rangle_E(t) - c^2t^2\right]E = 2c^2\left[\langle\mathbf{r}\rangle_P(t)\cdot\mathbf{P} - tE\right]. \tag{3.55}$$

According to (3.47), the combination appearing on the right is a constant of the motion, which we can put equal to zero by identifying the coordinate origin with $\langle\mathbf{r}\rangle_P$ at $t = 0$. The time integral of this equation is then

$$\langle r^2\rangle_E = (ct)^2 + \text{constant}, \tag{3.56}$$

which implies, for large times,

$$\left(\langle r^2\rangle_E(t)\right)^{1/2} \sim ct; \tag{3.57}$$

the center of energy of the pulse moves away from the origin at the speed of light.

What are the fields doing to enforce the conditions (3.51) of simple mechanical flow in a single direction? The relation between momentum and energy, (3.52), $E = |\mathbf{P}|c$, can be expressed in terms of the fields as

$$\int (d\mathbf{r})\,\frac{E^2+B^2}{8\pi} = \left|\int (d\mathbf{r})\,\frac{\mathbf{E}\times\mathbf{B}}{4\pi}\right|, \tag{3.58}$$

where the volume integrations are extended over the pulse. Now, a sum of vectors of given magnitudes is of maximum magnitude when all those vectors are parallel, which is to say here that

$$\int (d\mathbf{r})\,\frac{E^2+B^2}{2} \le \int (d\mathbf{r})\,|\mathbf{E}\times\mathbf{B}|, \tag{3.59}$$

where equality holds *only* when $\mathbf{E}\times\mathbf{B}$ everywhere points in the same direction, that of the pulse's total momentum or velocity. On the other hand, we note the inequality,

$$\begin{aligned}(\mathbf{E}\times\mathbf{B})^2 &= E^2B^2 - (\mathbf{E}\cdot\mathbf{B})^2 \\ &= \left(\frac{E^2+B^2}{2}\right)^2 - \left[\left(\frac{E^2-B^2}{2}\right)^2 + (\mathbf{E}\cdot\mathbf{B})^2\right] \le \left(\frac{E^2+B^2}{2}\right)^2,\end{aligned} \tag{3.60}$$

where the equality holds *only* if both $\mathbf{E}\cdot\mathbf{B} = 0$ and $E^2 = B^2$. So we deduce the opposite inequality to (3.59),

$$\int (d\mathbf{r})\,|\mathbf{E}\times\mathbf{B}| \le \int (d\mathbf{r})\,\frac{E^2+B^2}{2}. \tag{3.61}$$

Comparing (3.59) and (3.61), we see that both equalities must hold, so that

$$\mathbf{E}\cdot\mathbf{B} = 0, \qquad E^2 = B^2, \tag{3.62}$$

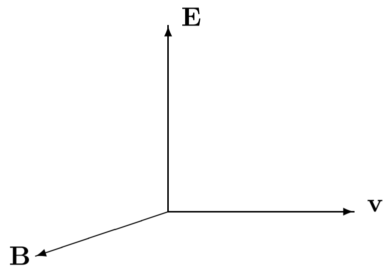

Figure 3.1: Electric and magnetic fields for an electromagnetic pulse propagating with velocity $\mathbf{v}$.

and $\mathbf{E}\times\mathbf{B}$ is unidirectional, pointing in the direction of propagation. Accordingly, the electric and magnetic fields in a unidirectional pulse are, everywhere within the pulse, of equal magnitude, mutually perpendicular, and perpendicular to the direction of motion of the pulse. (See Fig. 3.1.) These are the familiar properties of the electromagnetic fields of a light wave, which are here derived without recourse to explicit solutions to Maxwell's equations.

3.5 Problems for Chapter 3

1. The unit dyadic $\mathbf{1}$ is defined in terms of orthogonal unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ by

$$\mathbf{1} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk}.$$

Verify that ($\mathbf{A}$ is an arbitrary vector)

$$\mathbf{A}\cdot\mathbf{1} = \mathbf{A}, \quad \mathbf{1}\cdot\mathbf{A} = \mathbf{A}, \quad \mathbf{1}\cdot\mathbf{1} = \mathbf{1}.$$

Repeat, using components, i.e., $\mathbf{A}\cdot\mathbf{B} = A_iB_i$. Expand the following products of vectors with dyadics:

$$\mathbf{A}\cdot(\mathbf{BC}), \quad (\mathbf{AB})\cdot\mathbf{C}, \quad \mathbf{A}\times(\mathbf{BC}), \quad (\mathbf{AB})\times\mathbf{C}.$$

2. Let $\mathbf{A}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ be vector fields. Show that

$$\boldsymbol{\nabla}\cdot(\mathbf{AB}) = (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A}).$$

Let $\lambda(\mathbf{r})$ further be an arbitrary scalar function. Simplify

$$\boldsymbol{\nabla}\cdot(\lambda\mathbf{AB}), \quad \boldsymbol{\nabla}\cdot(\lambda\mathbf{A}\times\mathbf{B}).$$

3. An infinitesimal rotation is described by its effect on an arbitrary vector $\mathbf{V}$ by

$$\delta\mathbf{V} = \delta\boldsymbol{\omega}\times\mathbf{V},$$

where the direction of $\delta\boldsymbol{\omega}$ points in the direction of the rotation, and has magnitude equal to the (infinitesimal) amount of the rotation. Check that

$$\delta(\mathbf{V}^2) = 0.$$

The statement that, if $\mathbf{B}$ and $\mathbf{C}$ are vectors, so is $\mathbf{B}\times\mathbf{C}$, is expressed by

$$\delta(\mathbf{B}\times\mathbf{C}) = \delta\mathbf{B}\times\mathbf{C} + \mathbf{B}\times\delta\mathbf{C} = \delta\boldsymbol{\omega}\times(\mathbf{B}\times\mathbf{C}).$$

Verify directly the resulting relation among the arbitrary vectors.

4. Verify the following relations for the electromagnetic stress tensor:

   (a)
   $$\operatorname{Tr} T = T_{kk} = U,$$

   (b)
   $$\operatorname{Tr} T^2 = T_{kl}T_{lk} = 3U^2 - 2(c\mathbf{G})^2 \geq U^2,$$

   and

   (c)
   $$\det T = -U[U^2 - (c\mathbf{G})^2].$$

   Here the summation convention is employed, and the trace and determinant refer to T thought of as a 3×3 matrix.

5. Show that the angular momentum conservation law derived in Section 3.3 can be written as
   $$\frac{\partial}{\partial t}\boldsymbol{\mathcal{J}} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{K}} + \mathbf{r}\times\mathbf{f} = \mathbf{0},$$
   where the angular momentum density is
   $$\boldsymbol{\mathcal{J}} = \mathbf{r}\times\mathbf{G},$$
   and the angular momentum flux tensor is
   $$\boldsymbol{\mathcal{K}} = -\mathbf{T}\times\mathbf{r},$$
   the cross product referring to the second vector index of $\mathbf{T}$.

6. What if $\langle\mathbf{r}\rangle_P(0) \neq 0$ in (3.55)? Show that the integral of that equation can be interpreted in analogy with a group of particles that, at time $t = 0$, are set off with various positions and velocities, thereafter to move with those constant velocities,
   $$\mathbf{r}(t) = \mathbf{r}(0) + \mathbf{v}t.$$
   Square and average this position vector, and upon comparison with the solution of (3.55), identify $\langle\mathbf{r}(0)\cdot\mathbf{v}\rangle$ and v.

7. As in Problem 2.1, let

$$\mathbf{F} = \mathbf{E} + i\mathbf{B}, \quad \mathbf{F}^* = \mathbf{E} - i\mathbf{B}.$$

Identify the scalar

$$\frac{1}{8\pi}\mathbf{F}^* \cdot \mathbf{F},$$

the vector

$$\frac{1}{8\pi i}\mathbf{F}^* \times \mathbf{F},$$

and the dyadic

$$\frac{1}{8\pi}(\mathbf{F}\mathbf{F}^* + \mathbf{F}^*\mathbf{F}).$$

What happens to these quantities if $\mathbf{F}$ is replaced by $e^{-i\phi}\mathbf{F}$, ϕ being a constant?

8. Electric charge e is located at the fixed point $\frac{1}{2}\mathbf{R}$. Magnetic charge g is stationed at the fixed point $-\frac{1}{2}\mathbf{R}$. What is the momentum density at the arbitrary point $\mathbf{r}$? Verify that it is divergenceless by writing it as a curl. Evaluate the electromagnetic angular momentum, the integrated moment of the momentum density. Recognize that it is a gradient with respect to $\mathbf{R}$. Continue the evaluation to discover that it depends only on the direction of $\mathbf{R}$, not its magnitude. This is the naive, semiclassical basis for the charge quantization condition of Dirac,

$$eg = \frac{n}{2}\hbar c.$$