Chapter 1

Maxwell's Equations

The teaching of electromagnetic theory is something like that of American History in school; you get it again and again. Well, this is the end of the line. Here is where we put it all together, and yet, not quite, since it is still classical electrodynamics and the final goal is quantum electrodynamics. This preoccupation reflects the all-pervasive nature of electromagnetism, with implications ranging from the farthest galaxies to the interiors of the fundamental particles. In particular, the properties of ordinary matter, including those properties classified as chemical and biological, depend only on electromagnetic forces, in conjunction with the microscopic laws of quantum mechanics.

1.1 Electrostatics

Our intention is to move toward the general picture as quickly as possible, starting with a review of electrostatics. We take for granted the phenomenology of electric charge, including the Coulomb law of force between charges of dimensions that are small in comparison with their separation. This is expressed by the interaction energy, E, of a system of such charges in otherwise empty space, a vacuum:

$$E = \frac{1}{2} \sum_{\substack{a,b \\ a \neq b}} \frac{e_a e_b}{r_{ab}}, \tag{1.1}$$

where e_a is the charge of the ath particle while

$$r_{ab} = |\mathbf{r}_a - \mathbf{r}_b| \tag{1.2}$$

is the separation between the ath and bth particles. (Throughout this book we use the Gaussian system of units. Connection with the SI units will be given in Appendix A.) As we shall see, this starting point, the Coulomb energy (1.1), summarizes all the experimental facts of electrostatics. The energy of interaction of an individual charge with the rest of the system can be emphasized by rewriting (1.1) as

$$E = \frac{1}{2}\sum_a e_a \sum_{b\neq a} \frac{e_b}{r_{ab}} = \frac{1}{2}\sum_a e_a \phi_a, \tag{1.3}$$

where we have introduced the electrostatic potential at the location of the ath charge that is due to all the other charges,

$$\phi_a = \sum_{b\neq a} \frac{e_b}{r_{ab}}. \tag{1.4}$$

This is an action-at-a-distance point of view, in which the charge at a given point interacts with charges at other, distant points. Another approach, which generalizes and transcends action at a distance, employs the field concept (due to Faraday), a field being a local quantity, defined at every point of space. We take a first step in this direction by considering the potential as a field, which is defined everywhere, not just where the point charges are located. This generalized potential function, or simply the potential, $\phi(\mathbf{r})$, is

$$\phi(\mathbf{r}) = \sum_b \frac{e_b}{|\mathbf{r}-\mathbf{r}_b|}, \tag{1.5}$$

where we now treat every charge on an equal footing, which means that in (1.5) we sum over all charges e_b. In terms of this potential, which is different from ϕ_a, the energy E can be written as

$$E = \frac{1}{2}\sum_a e_a \phi(\mathbf{r}_a) - \sum_a E_a. \tag{1.6}$$

The last part of (1.6) is not to be understood numerically, but rather as an injunction to remove those terms in the first sum that refer to a single particle. In other words, we remove "self-action," leaving the mutual interactions between particles. The field concept naturally leads to self-action.

The notion of force is derived from that of energy, as we can see by considering the work done as a result of a spatial displacement. If we displace the ath charge by an amount $\delta\mathbf{r}_a$, the energy changes by an amount

$$\delta E = (\boldsymbol{\nabla}_a E)\cdot\delta\mathbf{r}_a = -\mathbf{F}_a\cdot\delta\mathbf{r}_a, \tag{1.7}$$

where $\mathbf{F}_a$ is the force acting on the ath point charge. Comparing this with the energy expression (1.1) we find the force on the ath particle to be

$$\mathbf{F}_a = -\boldsymbol{\nabla}_a \sum_{b\neq a}\frac{e_a e_b}{r_{ab}} = -\boldsymbol{\nabla}_a e_a \sum_{b\neq a}\frac{e_b}{r_{ab}} = -\boldsymbol{\nabla}_a e_a \phi(\mathbf{r}_a). \tag{1.8}$$

In the last form, we have substituted $\phi(\mathbf{r}_a)$ for ϕ_a, so it would appear that an extra self-action contribution has been introduced. To see that this is not true, we first argue physically that the difference between $\phi(\mathbf{r}_a)$ and ϕ_a is independent of position, and so self-action does not contribute to the force. Mathematically, what is this additional, unwanted, term? It is the negative gradient of the self-energy:

$$-\boldsymbol{\nabla} e_a \frac{e_a}{|\mathbf{r}-\mathbf{r}_a|}\bigg|_{\mathbf{r}\to\mathbf{r}_a} = e_a^2 \frac{\mathbf{r}-\mathbf{r}_a}{|\mathbf{r}-\mathbf{r}_a|^3}\bigg|_{\mathbf{r}\to\mathbf{r}_a}. \tag{1.9}$$

Can we make sense of this? We could define the limit here by arbitrarily adding a displacement vector $\boldsymbol{\epsilon}$ of fixed direction to $\mathbf{r}_a$ and letting its length approach zero:

$$\mathbf{r} = \mathbf{r}_a + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \to \mathbf{0}, \tag{1.10}$$

but at the cost of picking out a particular direction. In order to remove the most blatant aspect of this directional dependence, let us also approach $\mathbf{r}_a$ from the opposite direction,

$$\mathbf{r} = \mathbf{r}_a - \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \to \mathbf{0}, \tag{1.11}$$

and average over the two possibilities, so that the additional term (1.9) becomes

$$\longrightarrow e_a^2 \frac{1}{2}\left(\frac{\boldsymbol{\epsilon}}{|\boldsymbol{\epsilon}|^3} - \frac{\boldsymbol{\epsilon}}{|\boldsymbol{\epsilon}|^3}\right) = \mathbf{0}. \tag{1.12}$$

More elaborate limiting procedures, such as an average over all directions, can be used, but the simple procedure of (1.12) suffices. Therefore, we can employ $\phi(\mathbf{r})$ in (1.8), with the implicit use of the two-sided limit, (1.12), to calculate the force.

With the force given in terms of the gradient of a field (the potential), the electric field $\mathbf{E}$ can now be defined by

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}), \tag{1.13}$$

so that the force on a point charge e_a located at $\mathbf{r}_a$ is

$$\mathbf{F}_a = e_a \mathbf{E}(\mathbf{r}_a). \tag{1.14}$$

The electric field $\mathbf{E}$ so introduced is a function calculable at $\mathbf{r}$ in terms of the point charges located at $\mathbf{r}_b$,

$$\mathbf{E}(\mathbf{r}) = \sum_b e_b \frac{\mathbf{r}-\mathbf{r}_b}{|\mathbf{r}-\mathbf{r}_b|^3}. \tag{1.15}$$

As such, it remains an action-at-a-distance description, whereas, for many purposes, it would be much more convenient to be able to completely characterize the electric field by *local* properties. Such local statements will lead to differential equations, which, of course, must be supplemented by boundary conditions.

From its definition as the negative gradient of the potential, (1.13), the electric field has zero curl:

$$\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi(\mathbf{r}) = \mathbf{0}. \tag{1.16}$$

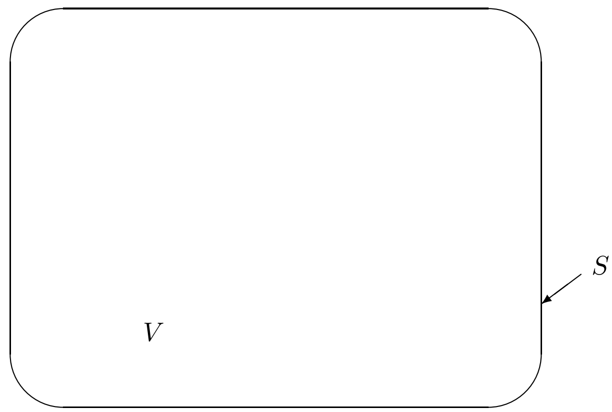

Figure 1.1: A surface S bounding a volume V used in computing the electric flux.

Besides the curl, the other elementary differential operation that can be applied to a vector field is the divergence. To find $\boldsymbol{\nabla}\cdot\mathbf{E}$, we consider a related integral statement. The integral of the normal component of $\mathbf{E}$ over a closed surface S bounding a volume V is the electric flux (see Fig. 1.1):

$$\oint_S d\mathbf{S}\cdot\mathbf{E}(\mathbf{r}) = \sum_b e_b \oint_S d\mathbf{S}\cdot\frac{\mathbf{r}-\mathbf{r}_b}{|\mathbf{r}-\mathbf{r}_b|}\frac{1}{|\mathbf{r}-\mathbf{r}_b|^2} = \sum_b e_b \oint_S d\Omega_b. \tag{1.17}$$

Here, $d\mathbf{S}$ is an area element, directed normal to the surface, and $d\Omega_b$ is an element of solid angle, which is defined in the following manner. The element of area perpendicular to the line from the bth charge is (see Fig. 1.2)

$$d\mathbf{S}\cdot\frac{\mathbf{r}-\mathbf{r}_b}{|\mathbf{r}-\mathbf{r}_b|}, \tag{1.18}$$

which, when divided by the square of the distance from the bth charge gives the solid angle $d\Omega_b$ subtended by $d\mathbf{S}$ as seen from the bth charge. There are now two possible situations: either e_b is inside, or it is outside the closed surface S, as shown in Fig. 1.3. Correspondingly, the integral over all solid angles in the two cases is

$$\oint_S d\Omega_b = \begin{cases} 4\pi & \text{if } e_b \text{ is inside } S, \\ 0 & \text{if } e_b \text{ is outside } S. \end{cases} \tag{1.19}$$

Hence, the electric flux through a closed surface S is proportional to the enclosed charge:

$$\oint_S d\mathbf{S}\cdot\mathbf{E}(\mathbf{r}) = \sum_{b\,\text{in}\,V} 4\pi e_b. \tag{1.20}$$

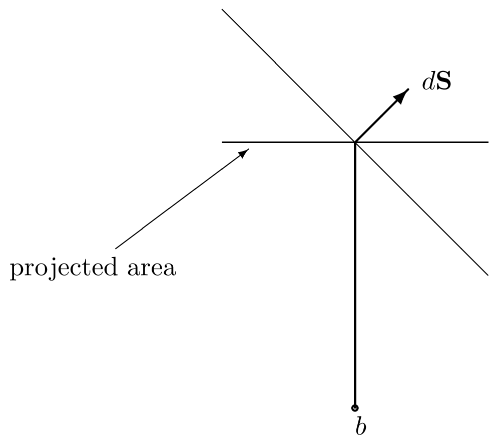

Figure 1.2: Geometrical definition of solid angle.

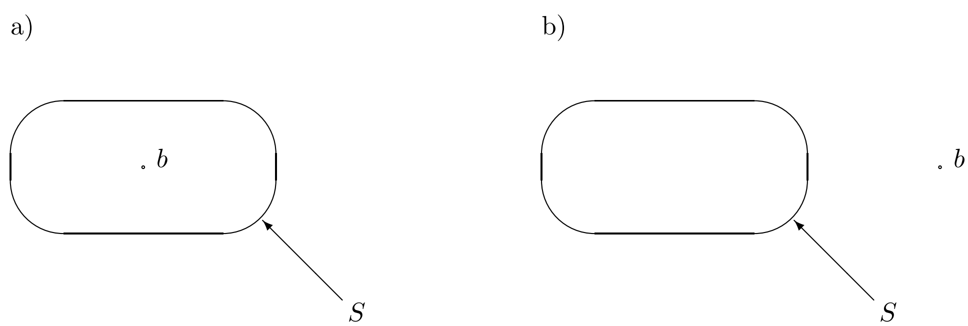

Figure 1.3: Topology if e_b is inside (a) or outside (b) the surface S.

This is the theorem of Carl Friedrich Gauss (1777–1855).

With our aim of deriving local statements in mind, we generalize the idea of point charges to that of a continuous distribution of charge, as measured by $\rho(\mathbf{r})$, the volume density of charge at the point $\mathbf{r}$. Then, the total charge in a volume V is obtained by integrating the charge density over that region:

$$\sum_{b\,\mathrm{in}\,V} e_b = \int_V (d\mathbf{r})\,\rho(\mathbf{r}). \tag{1.21}$$

[Throughout this book we use the following notation for the element of volume:

$$(d\mathbf{r}) = dx\,dy\,dz.] \tag{1.22}$$

For point charges, the charge density is zero except at the location of the charges,

$$\rho(\mathbf{r}) = \sum_b e_b \delta(\mathbf{r}-\mathbf{r}_b), \tag{1.23}$$

where the three-dimensional (Dirac) δ function is defined by

$$\int_V (d\mathbf{r})\,\delta(\mathbf{r}-\mathbf{r}_b) = \begin{cases} 0 \text{ if } \mathbf{r}_b \text{ is outside V,} \\ 1 \text{ if } \mathbf{r}_b \text{ is inside V.} \end{cases} \tag{1.24}$$

Then, the flux statement (1.20) becomes

$$4\pi \int_V (d\mathbf{r})\,\rho(\mathbf{r}) = \oint_S d\mathbf{S}\cdot\mathbf{E}(\mathbf{r}) = \int_V (d\mathbf{r})\,\boldsymbol{\nabla}\cdot\mathbf{E}(\mathbf{r}), \tag{1.25}$$

by use of the divergence theorem relating surface and volume integrals. (See Problem 1.2.) Since (1.25) is true for an arbitrary volume V, the integrands of the volume integrals must be equal, so we obtain the equation satisfied by the divergence of $\mathbf{E}$,

$$\boldsymbol{\nabla}\cdot\mathbf{E}(\mathbf{r}) = 4\pi\rho(\mathbf{r}). \tag{1.26}$$

These differential equations for the curl and divergence of $\mathbf{E}$, (1.16) and (1.26), respectively, completely characterize $\mathbf{E}$ when appropriate *boundary conditions* are imposed. It is evident from (1.15) that, for a localized charge distribution, the magnitude of the electric field becomes vanishingly small with increasing distance from the collection of charges:

$$|\mathbf{E}| \to 0 \quad \text{as} \quad \mathbf{r}\to\infty. \tag{1.27}$$

One can also specify how rapidly this occurs. But it is remarkable that the weak boundary condition (1.27) already implies a unique solution to the differential equations (1.16) and (1.26). To show this, we suppose that $\mathbf{E}_1$ and $\mathbf{E}_2$ are two such solutions. The difference, $\boldsymbol{\mathcal{E}} = \mathbf{E}_1 - \mathbf{E}_2$ satisfies

$$\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{E}} = 0, \quad \boldsymbol{\nabla}\times\boldsymbol{\mathcal{E}} = \mathbf{0} \quad \text{everywhere}, \tag{1.28}$$

$$|\boldsymbol{\mathcal{E}}| \to 0 \quad \text{as} \quad \mathbf{r}\to\infty, \tag{1.29}$$

from which we must prove that $\boldsymbol{\mathcal{E}} = \mathbf{0}$. The identity

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\mathcal{E}}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{E}}) - \nabla^2\boldsymbol{\mathcal{E}}, \tag{1.30}$$

combined with the vanishing of $\boldsymbol{\nabla}\times\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{E}}$, implies that

$$-\nabla^2\boldsymbol{\mathcal{E}} = \mathbf{0}. \tag{1.31}$$

Let the single function $\mathcal{E}(\mathbf{r})$ be any Cartesian component of the vector field $\boldsymbol{\mathcal{E}}$; it obeys

$$-\nabla^2\mathcal{E}(\mathbf{r}) = 0. \tag{1.32}$$

We present this as the everywhere valid statement

$$0 = -\mathcal{E}\nabla^2\mathcal{E} = -\boldsymbol{\nabla}\cdot(\mathcal{E}\boldsymbol{\nabla}\mathcal{E}) + (\boldsymbol{\nabla}\mathcal{E})^2, \tag{1.33}$$

or

$$(\boldsymbol{\nabla}\mathcal{E})^2 - \nabla^2\frac{1}{2}\mathcal{E}^2 = 0. \tag{1.34}$$

Now we integrate this over the interior volume $V(R)$ of a sphere of radius R centered about an arbitrary point, which we take as the origin. The integral of the second term in (1.34) is turned into an integral over the surface $S(R)$ of the sphere by means of the divergence theorem,

$$-\int_{V(R)}(d\mathbf{r})\,\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\frac{1}{2}\mathcal{E}^2) = -\oint_{S(R)} d\mathbf{S}\cdot\boldsymbol{\nabla}\frac{1}{2}\mathcal{E}^2 = -\oint_{S(R)} dS\,\frac{\partial}{\partial R}\frac{1}{2}\mathcal{E}^2. \tag{1.35}$$

Using the relation between an element of area and an element of solid angle, $dS = R^2 d\Omega$, we can present this surface integral in terms of the average value of $\mathcal{E}^2$ over the surface of the sphere,

$$\langle\mathcal{E}^2\rangle_R = \frac{1}{4\pi}\oint d\Omega\,\mathcal{E}^2. \tag{1.36}$$

And so the integral of (1.34) is

$$\int_{V(R)}(d\mathbf{r})\,(\boldsymbol{\nabla}\mathcal{E})^2 - 4\pi R^2\frac{d}{dR}\frac{1}{2}\langle\mathcal{E}^2\rangle_R = 0. \tag{1.37}$$

The decisive step now is to divide by the area $4\pi R^2$, and then integrate (1.37) over R from 0 to ∞:

$$\int_0^\infty dR\frac{1}{4\pi R^2}\int_{V(R)}(d\mathbf{r})\,(\boldsymbol{\nabla}\mathcal{E})^2 + \frac{1}{2}\langle\mathcal{E}^2\rangle_0 = 0, \tag{1.38}$$

which finally incorporates the boundary condition (1.29), that $\mathcal{E}$ vanishes at all infinitely remote points. Everything on the left side of (1.38) is non-negative, yet it all adds up to zero. Accordingly, every individual contribution must be zero. This tells us quite explicitly that $\mathcal{E} = 0$ at the origin, which is anywhere, and, consistently, that $\boldsymbol{\nabla}\mathcal{E} = 0$ everywhere, or, that $\mathcal{E}$ is a constant, which is required to be zero by the boundary condition. This being true of any component, we conclude that the vector $\boldsymbol{\mathcal{E}} = \mathbf{0}$. This completes our proof of the "uniqueness theorem" of electrostatics, that the differential equations (1.16) and (1.26) have a unique solution when the boundary condition (1.27) is imposed. (See Problem 1.3.)

From the Coulomb energy, we have thus derived the equations of *electrostatics*:

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{E} &= 4\pi\rho, \quad & \frac{\partial}{\partial t}\rho = 0, \\ \boldsymbol{\nabla}\times\mathbf{E} &= \mathbf{0}, \quad & \frac{\partial}{\partial t}\mathbf{E} = \mathbf{0}, \end{aligned} \tag{1.39}$$

where the time independence has been made explicit. We are now going to remove the restriction to static conditions by letting the charges move in a particularly simple way. The equations of electromagnetism that emerge from this discussion will then be accepted as applicable to more general motions, as justified by various tests of internal consistency.

1.2 Inference of Maxwell's Equations

We introduce time dependence in the simplest way by assuming that all charges are in uniform motion with a common velocity $\mathbf{v}$ as produced by transforming a static arrangement of charges to a coordinate system moving with velocity $-\mathbf{v}$. (We insist that the same physics applies in the two situations.) At first we will take $|\mathbf{v}|$ to be very small in comparison with a critical speed c, which will be identified with the speed of light. To catch up with the moving charges, one would have to move with their velocity, $\mathbf{v}$. Accordingly, the time derivative in the co-moving coordinate system, in which the charges are at rest, is the sum of explicit time dependent and coordinate dependent contributions,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}, \tag{1.40}$$

so, in going from the static system to the uniformly moving system, we make the replacement

$$\frac{\partial}{\partial t} \to \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}. \tag{1.41}$$

The equation for the constancy of the charge density in (1.39) becomes, in the moving system

$$0 = \frac{\partial\rho}{\partial t} \to \frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\rho, \tag{1.42}$$

or, since $\mathbf{v}$ is constant,

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\mathbf{v}\rho) = 0. \tag{1.43}$$

We recognize here a particular example of the charge flux vector or the (electric) current density $\mathbf{j}$,

$$\mathbf{j} = \rho\mathbf{v}. \tag{1.44}$$

The relation between charge density and current density,

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t)+\boldsymbol{\nabla}\cdot\mathbf{j}(\mathbf{r},t)=0, \tag{1.45}$$

is the general statement of the *conservation of charge*. Conservation demands that the rate of decrease of the charge within an arbitrary volume V must equal the rate at which the charge flows out of the bounding surface S, that is

$$-\frac{d}{dt}\int_V (d\mathbf{r})\,\rho(\mathbf{r},t)=\oint_S d\mathbf{S}\cdot\mathbf{j}(\mathbf{r},t)=\int_V (d\mathbf{r})\,\boldsymbol{\nabla}\cdot\mathbf{j}(\mathbf{r},t). \tag{1.46}$$

Since V is arbitrary, the local conservation law, (1.45), follows. We also note that the expression for the current density, (1.44), continues to be valid even when $\mathbf{v}$ is dependent upon position, $\mathbf{v}\to\mathbf{v}(\mathbf{r},t)$. (See Problem 1.4.)

We can perform a similar transformation on the equation for the electric field $\partial\mathbf{E}/\partial t=\mathbf{0}$; namely,

$$\mathbf{0}=\frac{d}{dt}\mathbf{E}=\frac{\partial\mathbf{E}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{E}. \tag{1.47}$$

Making use of a vector identity, together with (1.26) and (1.44), ($\mathbf{v}$ is constant),

$$\begin{aligned}\boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{E}) &= \mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{E})-(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{E} && (1.48)\\ &= \mathbf{v}4\pi\rho-(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{E} \\ &= 4\pi\mathbf{j}-(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{E}, && (1.49)\end{aligned}$$

we find an equation relating $\mathbf{E}$ to the current density,

$$\mathbf{0}=\frac{\partial\mathbf{E}}{\partial t}+4\pi\mathbf{j}-\boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{E}). \tag{1.50}$$

[Notice that by taking the divergence of (1.50) we recover the local charge conservation equation (1.45), so that the conservation of charge is not an independent statement.] The quantity $\mathbf{v}\times\mathbf{E}$ represents a new phenomenon combining the effects of motion with those of electric charge. To describe this new, induced effect, we define the magnetic induction[1] $\mathbf{B}$ by

$$\mathbf{v}\times\mathbf{E}=c\mathbf{B}, \tag{1.51}$$

where c is a constant having the dimensions of velocity (which will turn out to be the speed of light). Expressed in terms of the magnetic field, (1.50) becomes an equation determining the curl of $\mathbf{B}$,

$$\boldsymbol{\nabla}\times\mathbf{B}=\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}+\frac{4\pi}{c}\mathbf{j}. \tag{1.52}$$

Next, we naturally ask for the divergence of $\mathbf{B}$. According to the definition, (1.51), we have

$$\boldsymbol{\nabla}\cdot\mathbf{B}=\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{v}}{c}\times\mathbf{E}\right)=-\left(\frac{\mathbf{v}}{c}\times\boldsymbol{\nabla}\right)\cdot\mathbf{E}=-\frac{\mathbf{v}}{c}\cdot(\boldsymbol{\nabla}\times\mathbf{E})=0, \tag{1.53}$$

[1] We will usually call $\mathbf{B}$ the magnetic *field*, but see Chapter 4.

or

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0. \tag{1.54}$$

Moreover, in the co-moving coordinate system where the charges are at rest—static—the magnetic field should also not change in time:

$$\frac{d}{dt}\mathbf{B} = \frac{\partial}{\partial t}\mathbf{B} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{B} = \mathbf{0}, \tag{1.55}$$

which becomes, when we use the identity in (1.48) as well as (1.54),

$$\frac{\partial\mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B}), \tag{1.56}$$

consistent with $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$.

What do we do now? We need one experimental fact. Light is an electromagnetic oscillation. The evidence for this is overwhelming. As examples, we note that electric and magnetic fields are known to influence the emission, propagation, and absorption of light; and that radio and infrared waves, which differ only in wavelength from visible light, are emitted by electric charge oscillations. What must be done so that this fact is built into the equations we are inferring? The existence of electromagnetic waves means that the equations determining the electric field have solutions of the form

$$\mathbf{E} \sim \mathbf{f}(z - ct), \tag{1.57}$$

where c is the speed of the waves. Such waves, propagating in the z direction, satisfy the second-order differential equation

$$\frac{\partial^2}{\partial z^2}\mathbf{E} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}; \tag{1.58}$$

for an arbitrary direction of propagation, the corresponding wave equation is

$$\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}. \tag{1.59}$$

More precisely, we require that this equation should hold far from the charges that produce the field. The left side of this equation can be written as [cf. (1.30)]

$$\nabla^2\mathbf{E} = -\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}), \tag{1.60}$$

since $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ outside the charge distribution, while, by means of (1.52) and (1.56), the right side becomes ($\mathbf{j}$ is zero outside the charge distribution)

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{\nabla}\times\mathbf{B} = \frac{1}{c}\boldsymbol{\nabla}\times[\boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B})]. \tag{1.61}$$

This shows that the desired differential equation will hold if

$$\mathbf{E} = -\frac{\mathbf{v}}{c}\times\mathbf{B}. \tag{1.62}$$

But this cannot be a completely correct statement, since then $\mathbf{v} \to \mathbf{0}$ would require $\mathbf{E} \to \mathbf{0}$. No electrostatics! However, all that is really necessary is that the curl of this tentative identification be valid:

$$\boldsymbol{\nabla}\times\mathbf{E} = -\boldsymbol{\nabla}\times\left(\frac{\mathbf{v}}{c}\times\mathbf{B}\right), \tag{1.63}$$

or, if we use (1.56),

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}. \tag{1.64}$$

This is consistent with electrostatics since it generalizes $\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}$ to the time-dependent situation. The fact that $\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}$ has been used before to derive $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ is consistent here since the error is now seen to be of order $(v/c)^2$. [See (1.53).]

Collecting the above relations, you will recognize that we have arrived at Maxwell's equations,

$$\begin{aligned} \boldsymbol{\nabla}\times\mathbf{B} &= \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} + \frac{4\pi}{c}\mathbf{j}, \qquad \boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho, \\ -\boldsymbol{\nabla}\times\mathbf{E} &= \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}, \qquad\qquad \boldsymbol{\nabla}\cdot\mathbf{B} = 0. \end{aligned} \tag{1.65}$$

These equations of *electromagnetism*, as local, differential field equations, are no longer restricted to the initial assumption of a common small velocity for all charges.

To complete the dynamical picture we ask: What replaces (1.14) to describe the force on an electric charge, when that charge moves with some velocity $\mathbf{v}$ in given electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$? We consider two coordinate systems, one in which the particle is at rest (co-moving coordinate system) and one in which it moves at velocity $\mathbf{v}$. Suppose in the latter coordinate system, the electric and magnetic fields are given by $\mathbf{E}$ and $\mathbf{B}$, respectively. In the co-moving frame, the force on the particle is

$$\mathbf{F} = e\mathbf{E}_{\text{eff}}, \tag{1.66}$$

where $\mathbf{E}_{\text{eff}}$ is the electric field in this frame. In transforming to the co-moving frame, all the other charges—those responsible for $\mathbf{E}$ and $\mathbf{B}$—have been given an additional counter velocity $-\mathbf{v}$. We then infer from (1.62) that $(\mathbf{v}/c)\times\mathbf{B}$ has the character of an additional electric field in the co-moving frame. Hence, the suggested $\mathbf{E}_{\text{eff}}$ is

$$\mathbf{E}_{\text{eff}} = \mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}, \tag{1.67}$$

leading to the force law, due to Hendrick Antoon Lorentz (1835–1928),

$$\mathbf{F} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right). \tag{1.68}$$

These results, Maxwell's equations, (1.65), and the Lorentz force law, (1.68), have *not* been *derived*, but *inferred* from a special circumstance. We will adopt these equations as describing the electromagnetic fields produced by, and acting on, charges possessing arbitrary velocities, although the above discussion does allow room for additional terms if v/c is no longer small. The fact that no such terms are actually required is part of the implication of the special theory of relativity (see problem 1.6). We will prefer, instead, to show the physical consistency of the equations as they stand (see Chapter 3).

1.3 Discussion

We have arrived at the Maxwell-Lorentz electrodynamics by combining three ingredients: the laws of electrostatics; the Galileo-Newton principle of relativity (charges at rest, and charges with a common velocity viewed by a co-moving observer, are physically indistinguishable); and the existence of electromagnetic waves that travel in a vacuum at the speed c. The historical line of development was otherwise. Until the beginning of the nineteenth century, electricity and magnetism were unrelated phenomena. The discovery in 1820 by Hans Christian Oersted (1777–1851) that an electric current influences a magnet—creates a magnetic field—is formulated, for stationary currents, in the field equation

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j}. \tag{1.69}$$

The symbol c that appears in this equation is the ratio of electromagnetic and electrostatic units of electricity (see Appendix A). Then, in 1831, Michael Faraday (1791–1867) discovered that relative motion of a wire and a magnet induces a voltage in the wire—creates an electric field. Such is the content of

$$-\boldsymbol{\nabla}\times\mathbf{E} = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}, \tag{1.70}$$

which extends the magnetostatic relation

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \tag{1.71}$$

that expresses the empirical absence of single magnetic poles. Finally, in 1864, James Clerk Maxwell (1831–1879) recognized that the restriction to stationary currents in (1.69), as expressed by $\boldsymbol{\nabla}\cdot\mathbf{j} = 0$, was removed in

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}, \tag{1.72}$$

when joined to the electrostatic equation

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho. \tag{1.73}$$

The deduction of the existence of electromagnetic waves that travel at the speed c, in remarkable numerical agreement with the speed of light, was confirmed in 1867 by Heinrich Rudolf Hertz (1857–1894). It was the conflict between the existence of this absolute speed c and the relativity concept of Newtonian mechanics that set the stage for Einsteinian relativity. Already at the age of 16, Albert Einstein (1879–1955) had recognized this paradox: To a co-moving Newtonian observer, light waves should oscillate in space, but not move; however, Maxwell's equations admit no such solutions. Einsteinian relativity is an outgrowth of Maxwellian electrodynamics, not the other way about. That is the spirit in which electrodynamics is developed as a self-contained subject in this book.

1.4 Problems for Chapter 1

1. Verify the following identities explicitly:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})+\mathbf{B}\times(\mathbf{C}\times\mathbf{A})+\mathbf{C}\times(\mathbf{A}\times\mathbf{B})=\mathbf{0},$$

$$\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B})=\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B})-\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A})-(\mathbf{A}\times\boldsymbol{\nabla})\times\mathbf{B}+(\mathbf{B}\times\boldsymbol{\nabla})\times\mathbf{A},$$

$$\boldsymbol{\nabla}\cdot(\lambda\mathbf{A}\times\mathbf{B})=\lambda(\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{A}-\mathbf{A}\cdot\boldsymbol{\nabla}\times\mathbf{B})+\mathbf{A}\times\mathbf{B}\cdot\boldsymbol{\nabla}\lambda.$$

2. Verify, using Cartesian coordinates, the divergence theorem,

$$\int_V (d\mathbf{r})\,\boldsymbol{\nabla}\cdot\mathbf{E}=\oint_S d\mathbf{S}\cdot\mathbf{E},$$

where V is the volume contained within the closed surface S, $d\mathbf{S}$ being the surface element in the direction of the outward normal, and Stokes' theorem,

$$\int_S d\mathbf{S}\cdot(\boldsymbol{\nabla}\times\mathbf{E})=\oint_C d\mathbf{l}\cdot\mathbf{E},$$

where C is the closed boundary of the open surface S, and $d\mathbf{l}$ is the tangentially directed line element. The sense of the line integration is given by the right hand rule. [That is, if the contour C is traversed in the sense of the fingers of the right hand, the thumb points in the sense of the orientation of the surface.]

3. This question has to do with the uniqueness theorem which follows from (1.37).

   (a) Directly from that equation, what assumption about $|\mathcal{E}(\mathbf{r})|$, $|\mathbf{r}|\to\infty$, will produce the conclusion that $\mathcal{E}=0$ everywhere?

   (b) How would it work out if one had integrated this equation from $R=0$ to ∞, without dividing by R^2?

   (c) How fast would $\langle\mathcal{E}^2\rangle_R$ have to fall off with R so that we could conclude $\mathcal{E}=0$ everywhere by simply taking $R\to\infty$ in (1.37)?

4. For an arbitrarily moving charge, the charge and current densities are

$$\rho(\mathbf{r},t) = e\delta(\mathbf{r}-\mathbf{R}(t)), \quad \mathbf{j}(\mathbf{r},t) = e\frac{d\mathbf{R}}{dt}\delta(\mathbf{r}-\mathbf{R}(t)),$$

where $\mathbf{R}(t)$ is the position vector of the charged particle. Verify the statement of conservation of charge,

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t) + \boldsymbol{\nabla}\cdot\mathbf{j}(\mathbf{r},t) = 0.$$

5. In a region where no charges are present, the potential satisfies Laplace's equation,

$$\nabla^2\phi = 0.$$

Such a function is called harmonic. Show that in a region where the potential is harmonic, the potential nowhere assumes a maximum or minimum value. Use this result to give another proof of the uniqueness theorem of electrostatics proved in Section 1.1.

6. In this chapter we "derived" Maxwell's equations by exploiting approximate Galilean invariance. However, we cannot push Galilean invariance further, since it is not valid in $O(v^2/c^2)$. The correct relativity is that of Einstein. Verify that Maxwell's equations are invariant under the transformations of Einstein's special relativity, as follows. Consider a Lorentz transformation corresponding to a boost in the x direction, which on the space-time coordinates is defined by

$$\begin{aligned} x_0' &= \gamma(x_0 - \beta x_1), \\ x_1' &= \gamma(x_1 - \beta x_0), \\ x_2' &= x_2, \\ x_3' &= x_3. \end{aligned}$$

Here $x_0 = ct$, $x_1 = x$, $x_2 = y$, $x_3 = z$, and

$$\beta = \frac{v}{c}, \quad \gamma = (1-\beta^2)^{-1/2},$$

$\mathbf{v}$ being the relative velocity of the two coordinate frames. We can regard the four quantities x_μ, $\mu = 0$, 1, 2, 3, as forming a four-vector. The four-current j_μ, $j_0 = c\rho$, j_i, $i = 1$, 2, 3, constructed from the electric charge and current densities, transforms by the same law:

$$\begin{aligned} j_0' &= \gamma(j_0 - \beta j_1), \\ j_1' &= \gamma(j_1 - \beta j_0), \\ j_2' &= j_2, \\ j_3' &= j_3. \end{aligned}$$

On the other hand, the electric and magnetic field vectors are components of a four-tensor, and so they have a more complicated transformation law. Consider a boost by an arbitrary velocity $\mathbf{v}$. Then the components of the electric and magnetic fields in the direction of $\mathbf{v}$ do not change, while the components in directions perpendicular to $\mathbf{v}$ are entangled:

$$\begin{aligned} \mathbf{E}'_{\|} &= \mathbf{E}_{\|}, \quad \mathbf{E}'_{\perp} = \gamma \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right)_{\perp}, \\ \mathbf{B}'_{\|} &= \mathbf{B}_{\|}, \quad \mathbf{B}'_{\perp} = \gamma \left(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E} \right)_{\perp}. \end{aligned}$$

For $\mathbf{v} = (v, 0, 0)$ verify explicitly that if Maxwell's equations hold in the unprimed frame, they hold in the primed frame as well, no matter how near v may approach c. This was essentially the path by which Lorentz and Poincaré derived the transformation equations (but not the physics) of special relativity. A more complete treatment of Einsteinian relativity will be given in Chapter 10.

7. A charge e moves in the vacuum under the influence of uniform fields $\mathbf{E}$ and $\mathbf{B}$. Assume that $\mathbf{E} \cdot \mathbf{B} = 0$ and that $\mathbf{v} \cdot \mathbf{B} = 0$. At what velocity does the charge move without acceleration? What is its speed when $|\mathbf{E}| = |\mathbf{B}|$?

Chapter 2

Magnetic Charge I

Our discussion in Chapter 1 contains a certain implicit assumption. When it came to (1.62),

$$\mathbf{E} = -\frac{\mathbf{v}}{c}\times\mathbf{B}, \tag{2.1}$$

with its implication that static electric charges produce no electric field, we knew better than to accept this and altered it to

$$\boldsymbol{\nabla}\times\mathbf{E} = -\boldsymbol{\nabla}\times\left(\frac{\mathbf{v}}{c}\times\mathbf{B}\right), \tag{2.2}$$

thereby admitting, for $\mathbf{v} = \mathbf{0}$, a static electric field, one obeying $\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}$. Why then did we earlier accept without question the relation (1.51),

$$\mathbf{B} = \frac{\mathbf{v}}{c}\times\mathbf{E}, \tag{2.3}$$

with its implication that all magnetic fields are due to the motion of electric charges? This is the (1820) hypothesis of André Marie Ampère (1775–1836). But is it true? An affirmative response is conventional, but the mathematical development allows a more general possibility. Again, all that was really required in the above was the curl relation

$$\boldsymbol{\nabla}\times\mathbf{B} = \boldsymbol{\nabla}\times\left(\frac{\mathbf{v}}{c}\times\mathbf{E}\right), \tag{2.4}$$

admitting the possibility, for $\mathbf{v} = \mathbf{0}$, of a static magnetic field obeying $\boldsymbol{\nabla}\times\mathbf{B} = \mathbf{0}$, one that has its origin in *magnetic* charge. If ρ_m is the density of such magnetic charge, the analogy with electrostatics suggests that

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 4\pi\rho_m. \tag{2.5}$$

The implication of (2.5) is that a further source of magnetic fields, other than moving electric charges, could exist in magnetic charge. Whether this possibility is realized in nature still awaits experimental confirmation.

Further changes in Maxwell's equations are required if magnetic charge exists. Since then $\boldsymbol{\nabla}\cdot\mathbf{B} \neq 0$, the co-moving time derivative of $\mathbf{B}$ becomes [*cf.* (1.48)]

$$\mathbf{0} = \frac{d}{dt}\mathbf{B} = \frac{\partial}{\partial t}\mathbf{B} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{B} = \frac{\partial}{\partial t}\mathbf{B} - \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B}) + \mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{B}). \tag{2.6}$$

Then, using (2.5) and the magnetic current density $\mathbf{j}_m$, defined as

$$\mathbf{j}_m = \rho_m \mathbf{v}, \tag{2.7}$$

together with (2.2), we obtain the following modified Maxwell equation

$$-\boldsymbol{\nabla}\times\mathbf{E} = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} + \frac{4\pi}{c}\mathbf{j}_m. \tag{2.8}$$

Notice that (2.8) implies the conservation of magnetic charge:

$$\frac{\partial}{\partial t}\rho_m + \boldsymbol{\nabla}\cdot\mathbf{j}_m = 0. \tag{2.9}$$

The complete set of Maxwell's equations, when magnetic charge is present, now reads

$$\begin{aligned} \boldsymbol{\nabla}\times\mathbf{B} &= \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} + \frac{4\pi}{c}\mathbf{j}_e, \qquad \boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho_e, \\ -\boldsymbol{\nabla}\times\mathbf{E} &= \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} + \frac{4\pi}{c}\mathbf{j}_m, \qquad \boldsymbol{\nabla}\cdot\mathbf{B} = 4\pi\rho_m, \end{aligned} \tag{2.10}$$

where we have consistently used the subscript e to denote densities for electric charge. Observe that these equations are invariant in form under the replacements (duality transformation)

$$\begin{aligned} \rho_e &\to \rho_m, \qquad \mathbf{E} \to \mathbf{B}, \qquad \mathbf{j}_e \to \mathbf{j}_m, \\ \rho_m &\to -\rho_e, \qquad \mathbf{B} \to -\mathbf{E}, \qquad \mathbf{j}_m \to -\mathbf{j}_e. \end{aligned} \tag{2.11}$$

The generalized Lorentz force law, suggested by this symmetry, is

$$\mathbf{F} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right) + g\left(\mathbf{B} - \frac{\mathbf{v}}{c}\times\mathbf{E}\right), \tag{2.12}$$

for a particle carrying both electric and magnetic charge, e and g, respectively.

Although from time to time there have been spectacular reports of the discovery of magnetic charge (Price, 1975; Cabrera, 1982), these "discoveries" were never replicated, and serious objections were raised in each instance. Nevertheless, there are strong theoretical reasons to believe that magnetic charge exists in nature, and may have played an important role in the development of the universe. Searches for magnetic charge continue at the present time, emphasizing that electromagnetism is very far from being a closed subject.

2.1 A Very Brief History of Magnetic Charge

It is said that Peregrinus in 1269 observed that magnets (lodestones) always have two poles, which he called north and south. This was elevated to a "hypothesis" by Ampère in the early 19th Century. The first theoretical calculation of the motion of a charged particle in the presence of a single magnetic pole was performed by Poincaré in 1896 to explain recent observations. A few years later, Thomson showed that a static system consisting of a magnetic pole and an electric charge possessed an angular momentum—see Problem 3.8. It was Dirac in 1931 who showed that magnetic charge was consistent with quantum mechanics only if electric and magnetic charges were quantized: For a system consisting of a pure magnetic charge g and a pure electric charge e, eg had to be an integral (or half-integral) multiple of $\hbar c$. Many people have contributed to the theory of magnetic charge subsequently; notable is the work of Schwinger in the 1960s and 1970s, especially his concept of dyons, particles which carry both electric and magnetic charge.

Many searches, both terrestrial and cosmic, have been carried out to find magnetic monopoles in nature, but, so far, to no avail. Worth mentioning is the induction technique of Luis Alvarez, *et al.* Positive reports were given by Price in 1975 [cited in the *Reader's Guide*] and by Blas Cabrera in 1982.[1] These, however, were never confirmed, and are no longer believed to offer any evidence for magnetic charge, even by their authors.

However, modern unified theories of fundamental interactions typically imply the existence of magnetic monopoles, or of dyons, often at extremely high mass scales ($\sim 10^{16}$ GeV), but perhaps at nearly accessible energies (~ 10 TeV). Moreover, there appears to be no reason why an elementary monopole or dyon of the Dirac-Schwinger type could not exist. So experimental searches continue.

2.2 Problems for Chapter 2

1. Write Maxwell's equations with magnetic charge in terms of
$$\mathbf{F} = \mathbf{E} + i\mathbf{B}, \quad i = \sqrt{-1},$$
and related combinations of charge and current. Verify that these equations retain their form under the transformation illustrated by
$$\mathbf{F} \to e^{-i\phi}\mathbf{F},$$
where ϕ is an arbitrary constant. Express this as a transformation of $\mathbf{E}$, $\mathbf{B}$, and the charge-current quantities. What is the geometric interpretation? What is the particular form of this transformation when $\phi = \pi/2$?

2. Suppose every charged particle carried electric *and* magnetic charge in the universal ratio $g_k/e_k = \lambda$. Is there another way of looking at this situation in which we would be unaware of magnetic charge?

[1] B. Cabrera, *Phys. Rev. Lett.* **48**, 1378 (1982).