

Kimball A. Milton and Julian Schwinger

# Electromagnetic Radiation

Variational Methods, Waveguides, and  
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## Maxwell's Equations

### 1.1 Microscopic Electrodynamics

Electromagnetic phenomena involving matter in bulk are approximately described by the Maxwell field equations, in SI units,<sup>1</sup>

$$\nabla \times \mathbf{H} = \frac{\partial}{\partial t} \mathbf{D} + \mathbf{J}, \quad \nabla \cdot \mathbf{D} = \rho, \quad (1.1a)$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0, \quad (1.1b)$$

together with constitutive equations of the medium which in their most common form are

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}, \quad \mathbf{J} = \sigma \mathbf{E}. \quad (1.2)$$

This theory takes no cognizance of the atomic structure of matter, but rather regards matter as a continuous medium that is completely characterized by the three constants  $\varepsilon$ ,  $\mu$ , and  $\sigma$ . Here  $\varepsilon$  is the electric permittivity (or “dielectric constant”),  $\mu$  is the magnetic permeability, and  $\sigma$  is the electric conductivity. The dependence of these material parameters on the nature of the substance, density, temperature, oscillation frequency, and so forth, is to be determined empirically. Opposed to this point of view, which we shall call macroscopic, is that initiated by Lorentz as an attempt to predict the properties of gross matter from the postulated behavior of atomic constituents. It is the two-fold purpose of such a theory to deduce the Maxwell equations as an approximate consequence of more fundamental microscopic field equations and to relate the macroscopic parameters  $\varepsilon$ ,  $\mu$ , and  $\sigma$  to atomic properties. Although the macroscopic theory forms an entirely adequate basis for our work in this monograph, the qualitative information given by simple atomic models is of such value that we begin with an account of the microscopic theory.

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<sup>1</sup> See the Appendix for a discussion of the different unit systems still commonly employed for electromagnetic phenomena.

### 1.1.1 Microscopic Charges

That attribute of matter which interacts with an electromagnetic field is electric charge. Charge is described by two quantities, the charge density  $\rho(\mathbf{r}, t)$  and the current density  $\mathbf{j}(\mathbf{r}, t)$ . The charge density is defined by the statement that the total charge  $Q$ , within an arbitrary volume  $V$  at the time  $t$ , is represented by the volume integral [ $(d\mathbf{r}) = dx dy dz$  is the element of volume]

$$Q = \int_V (d\mathbf{r}) \rho(\mathbf{r}, t) . \quad (1.3)$$

Of particular interest is the point charge distribution which is such that the total charge in any region including a fixed point  $\mathbf{R}$  is equal to a constant  $q$ , independent of the size of the region, while the total charge in any region that does not include the point  $\mathbf{R}$  vanishes. The charge density of the point distribution will be written

$$\rho(\mathbf{r}) = q \delta(\mathbf{r} - \mathbf{R}) , \quad (1.4)$$

with the  $\delta$  function defined by the statements

$$\int_V (d\mathbf{r}) \delta(\mathbf{r} - \mathbf{R}) = \begin{cases} 1 , & \mathbf{R} \text{ within } V , \\ 0 , & \mathbf{R} \text{ not within } V . \end{cases} \quad (1.5)$$

It is a consequence of this definition that the  $\delta$  function vanishes at every point save  $\mathbf{R}$ , and must there be sufficiently infinite to make its volume integral unity. No such function exists, of course, but it can be approximated with arbitrary precision. We need only consider, for example, the discontinuous function defined by

$$\delta_\epsilon(\mathbf{r} - \mathbf{R}) = \begin{cases} 0 , & |\mathbf{r} - \mathbf{R}| > \epsilon , \\ \frac{1}{\frac{4}{3}\pi\epsilon^3} , & |\mathbf{r} - \mathbf{R}| < \epsilon , \end{cases} \quad (1.6)$$

in the limit as  $\epsilon \rightarrow 0$ . Other possible representations are

$$\delta(\mathbf{r} - \mathbf{R}) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi^2} \frac{\epsilon}{(|\mathbf{r} - \mathbf{R}|^2 + \epsilon^2)^2} , \quad (1.7a)$$

$$\delta(\mathbf{r} - \mathbf{R}) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^3} e^{-\pi|\mathbf{r} - \mathbf{R}|^2/\epsilon^2} . \quad (1.7b)$$

We shall not hesitate to treat the  $\delta$  function as an ordinary, differentiable function.

The elementary constituents of matter, which for our purposes may be considered to be electrons and atomic nuclei, can ordinarily be treated as point charges, for their linear dimensions ( $\sim 10^{-13}$  cm) are negligible in comparison with atomic distances ( $\sim 10^{-8}$  cm). The charge density of a number of point charges with charges  $q_a$  located at the points  $\mathbf{r}_a$ ,  $a = 1, \dots, n$  is

$$\rho(\mathbf{r}) = \sum_{a=1}^n q_a \delta(\mathbf{r} - \mathbf{r}_a) . \quad (1.8)$$

If the charges are in motion, the charge density will vary in time in consequence of the time dependence of  $\mathbf{r}_a(t)$ . The time derivative, for fixed  $\mathbf{r}$ , is

$$\frac{\partial}{\partial t} \rho(\mathbf{r}, t) = \sum_{a=1}^n q_a \mathbf{v}_a \cdot \nabla_{\mathbf{r}_a} \delta(\mathbf{r} - \mathbf{r}_a) = - \sum_{a=1}^n q_a \mathbf{v}_a \cdot \nabla_{\mathbf{r}} \delta(\mathbf{r} - \mathbf{r}_a) , \quad (1.9)$$

or

$$\frac{\partial}{\partial t} \rho(\mathbf{r}, t) + \nabla_{\mathbf{r}} \cdot \sum_{a=1}^n q_a \mathbf{v}_a \delta(\mathbf{r} - \mathbf{r}_a) = 0 , \quad (1.10)$$

where  $\mathbf{v}_a = \frac{d}{dt} \mathbf{r}_a$  is the velocity of the  $a$ th point charge.

Charge in motion constitutes a current. The current density or charge flux vector  $\mathbf{j}(\mathbf{r}, t)$  is defined by the equation

$$I = \int_S dS \mathbf{n} \cdot \mathbf{j}(\mathbf{r}, t) , \quad (1.11)$$

where  $I dt$  is the net charge crossing an arbitrary surface  $S$  in the time interval  $dt$ . Positive charge crossing the surface in the direction of the normal  $\mathbf{n}$ , or negative charge moving in the opposite direction, make a positive contribution to the total current  $I$ , while charges with the reversed motion from these are assigned negative weight factors in computing  $I$ . The total charge leaving an arbitrary region bounded by the closed surface  $S$ , in the time interval  $dt$ , is

$$dQ = dt \oint_S dS \mathbf{n} \cdot \mathbf{j}(\mathbf{r}, t) , \quad (1.12)$$

where  $\mathbf{n}$  is the outward-drawn normal to the surface  $S$ . The fundamental property of charge, indeed its defining characteristic, is indestructibility. Thus the net amount of charge that flows across the surface  $S$  bounding  $V$  must equal the loss of charge within the volume. Hence

$$\oint_S dS \mathbf{n} \cdot \mathbf{j}(\mathbf{r}, t) \equiv \int_V (d\mathbf{r}) \nabla \cdot \mathbf{j}(\mathbf{r}, t) = - \frac{\partial}{\partial t} \int_V (d\mathbf{r}) \rho(\mathbf{r}, t) , \quad (1.13)$$

in which we have also employed the divergence theorem relating surface and volume integrals. Since the statement must be valid for an arbitrary volume, we obtain as the conservation equation of electric charge

$$\nabla \cdot \mathbf{j}(\mathbf{r}, t) + \frac{\partial}{\partial t} \rho(\mathbf{r}, t) = 0 . \quad (1.14)$$

It will be noted that an equation of precisely this form has been obtained for an assembly of point charges in (1.10), with

$$\mathbf{j}(\mathbf{r}, t) = \sum_{a=1}^n q_a \mathbf{v}_a \delta(\mathbf{r} - \mathbf{r}_a) . \quad (1.15)$$

Thus, for a single point charge,

$$\mathbf{j} = \rho \mathbf{v} . \quad (1.16)$$

The elementary charged constituents of matter possess inertia. Associated with charges in motion, therefore, are the mechanical properties of kinetic energy, linear momentum, and angular momentum. The definitions of these quantities for a system of  $n$  particles with masses  $m_a$ ,  $a = 1, \dots, n$ , are, respectively,

$$E = \sum_{a=1}^n \frac{1}{2} m_a v_a^2 , \quad (1.17a)$$

$$\mathbf{p} = \sum_{a=1}^n m_a \mathbf{v}_a , \quad (1.17b)$$

$$\mathbf{L} = \sum_{a=1}^n m_a \mathbf{r}_a \times \mathbf{v}_a , \quad (1.17c)$$

provided all particle velocities are small in comparison with  $c$ , the velocity of light in vacuo. The more rigorous relativistic expressions are

$$E = \sum_{a=1}^n m_a c^2 \left( \frac{1}{\sqrt{1 - v_a^2/c^2}} - 1 \right) , \quad (1.18a)$$

$$\mathbf{p} = \sum_{a=1}^n \frac{m_a}{\sqrt{1 - v_a^2/c^2}} \mathbf{v}_a , \quad (1.18b)$$

$$\mathbf{L} = \sum_{a=1}^n \frac{m_a}{\sqrt{1 - v_a^2/c^2}} \mathbf{r}_a \times \mathbf{v}_a , \quad (1.18c)$$

but this refinement is rarely required in studies of atomic structure.

### 1.1.2 The Field Equations

The electromagnetic field is described by two vectors, the electric field intensity (or electric field strength)  $\mathbf{e}(\mathbf{r}, t)$  and the magnetic field intensity (or magnetic induction)  $\mathbf{b}(\mathbf{r}, t)$ . [In this Chapter, for pedagogical purposes, we will use lower-case letters to denote the microscopic fields, for which we will use (rationalized) Heaviside-Lorentz units. See the Appendix.] The equations defining these vectors in relation to each other and to the charge-current distribution are postulated to be

$$\nabla \times \mathbf{b} = \frac{1}{c} \frac{\partial}{\partial t} \mathbf{e} + \frac{1}{c} \mathbf{j}, \quad \nabla \cdot \mathbf{e} = \rho, \quad (1.19a)$$

$$\nabla \times \mathbf{e} = -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{b}, \quad \nabla \cdot \mathbf{b} = 0, \quad (1.19b)$$

which are known variously as the microscopic field equations, or the Maxwell-Lorentz equations. Correspondence is established with the physical world by the further postulate that an electromagnetic field possesses the mechanical attributes of energy and momentum. These quantities are considered to be spatially distributed in the field, and it is therefore necessary to introduce not only measures of density, analogous to the charge density, but in addition measures of flux, analogous to the current density. We define

- energy density:

$$U = \frac{e^2 + b^2}{2}, \quad (1.20a)$$

- energy flux vector or the Poynting vector:

$$\mathbf{S} = c \mathbf{e} \times \mathbf{b}, \quad (1.20b)$$

- linear momentum density:

$$\mathbf{G} = \frac{1}{c} \mathbf{e} \times \mathbf{b}, \quad (1.20c)$$

- linear momentum flux dyadic or the stress dyadic:

$$\mathbb{T} = \mathbf{1} \frac{e^2 + b^2}{2} - \mathbf{e}\mathbf{e} - \mathbf{b}\mathbf{b}, \quad (1.20d)$$

The symbol  $\mathbf{1}$  indicates the unit dyadic. The basis for these definitions are certain differential identities, valid in the absence of charge and current, which have the form of conservation equations, analogous to that for electric charge. It may be directly verified that ( $\rho = 0$ ,  $\mathbf{j} = \mathbf{0}$ )

$$\frac{\partial}{\partial t} U + \nabla \cdot \mathbf{S} = 0, \quad \frac{\partial}{\partial t} \mathbf{G} + \nabla \cdot \mathbb{T} = \mathbf{0}, \quad (1.21)$$

on employing the identities

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{A}, \quad (1.22a)$$

$$(\nabla \times \mathbf{A}) \times \mathbf{A} = -\mathbf{A} \times (\nabla \times \mathbf{A}) = (\mathbf{A} \cdot \nabla) \mathbf{A} - \nabla \frac{1}{2} A^2. \quad (1.22b)$$

The total energy

$$E = \int (\mathbf{dr}) U \quad (1.23)$$

and the total linear momentum,

$$\mathbf{p} = \int (\mathrm{d}\mathbf{r}) \mathbf{G} \quad (1.24)$$

of an electromagnetic field confined to a finite region of space, are constant in time, for no energy or momentum flows through a surface enclosing the entire field. Energy and momentum, like charge, are recognized by the property of permanence.

The relation between the energy and momentum quantities expressed by

$$\mathbf{S} = c^2 \mathbf{G} \quad (1.25)$$

is a consequence of, or at least is consistent with, the relativistic connection between energy and mass,

$$E = mc^2 . \quad (1.26)$$

This may be seen from the remark that the momentum density can also be considered a mass flux vector, or alternatively, by the following considerations. On multiplying the energy conservation equation in (1.21) by  $\mathbf{r}$  and rearranging terms, we obtain

$$\frac{\partial}{\partial t} \mathbf{r} U + \nabla \cdot (\mathbf{S} \mathbf{r}) = \mathbf{S} = c^2 \mathbf{G} , \quad (1.27)$$

which, on integration over a volume enclosing the entire field, yields

$$\mathbf{p} = \frac{\mathrm{d}}{\mathrm{d}t} \int (\mathrm{d}\mathbf{r}) \mathbf{r} \frac{U}{c^2} = \frac{E}{c^2} \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \frac{E}{c^2} \mathbf{V} , \quad (1.28)$$

where

$$\mathbf{R} = \frac{1}{E} \int (\mathrm{d}\mathbf{r}) \mathbf{r} U \quad (1.29)$$

is the energy center of gravity of the field, which moves with velocity  $\mathbf{V} = \mathrm{d}\mathbf{R}/\mathrm{d}t$ . We have here the conventional relation between momentum and velocity, with  $E/c^2$  playing the role of the total mass of the electromagnetic field.

The velocity of the energy center of gravity,  $\mathbf{V}$ , which we shall term the group velocity of the field, is necessarily less in magnitude than the velocity of light. This is a result of the identity

$$(\mathbf{e} \times \mathbf{b})^2 = \left( \frac{e^2 + b^2}{2} \right)^2 - \left( \frac{e^2 - b^2}{2} \right)^2 - (\mathbf{e} \cdot \mathbf{b})^2 , \quad (1.30)$$

and the consequent inequality

$$|\mathbf{e} \times \mathbf{b}| \leq \frac{e^2 + b^2}{2} , \quad (1.31)$$

for from (1.24)

$$|\mathbf{p}| \leq \frac{1}{c} \int (\mathrm{d}\mathbf{r}) |\mathbf{e} \times \mathbf{b}| \leq \frac{E}{c}, \quad (1.32)$$

and therefore

$$|\mathbf{V}| \leq c. \quad (1.33)$$

Equality of  $|\mathbf{V}|$  with  $c$  is obtained only when  $\mathbf{e} \cdot \mathbf{b} = 0$ ,  $e^2 = b^2$ , and  $\mathbf{e} \times \mathbf{b}$  has the same direction everywhere. That is, the electric and magnetic field intensities must be equal in magnitude, perpendicular to each other, and to a fixed direction in space, as is the case for an ideal plane wave. More generally, we call such a configuration a unidirectional light pulse, for which further properties are given in Problem 1.34.

Another velocity associated with the field can be defined in terms of the center of gravity of the momentum distribution. We proceed from the conservation of momentum equation in (1.21) written, for manipulatory convenience, in component form,

$$\frac{\partial}{\partial t} G_j + \sum_{i=1}^3 \frac{\partial}{\partial x_i} T_{ij} = 0, \quad j = 1, 2, 3. \quad (1.34)$$

On multiplying this equation by  $x_j$ , and summing with respect to the index  $j$ , we obtain

$$\frac{\partial}{\partial t} \sum_i x_i G_i + \sum_{i,j} \frac{\partial}{\partial x_i} (T_{ij} x_j) = \sum_i T_{ii} \equiv \mathrm{Tr} \mathbf{T}, \quad (1.35)$$

(which introduces the concept of the trace of the dyadic  $\mathbf{T}$ ,  $\mathrm{Tr} \mathbf{T}$ ) or, returning to vector notation,

$$\frac{\partial}{\partial t} (\mathbf{r} \cdot \mathbf{G}) + \nabla \cdot (\mathbf{T} \cdot \mathbf{r}) = U, \quad (1.36)$$

for (note that we do not use the summation convention over repeated indices here)

$$T_{ii} = U - (e_i^2 + b_i^2), \quad \mathrm{Tr} \mathbf{T} = U. \quad (1.37)$$

The relation (1.36) thus established between the energy density and momentum quantities we shall call the virial theorem. On integration over the entire region occupied by the field, we find

$$E = \frac{\mathrm{d}}{\mathrm{d}t} \int (\mathrm{d}\mathbf{r}) \mathbf{r} \cdot \mathbf{G} \equiv \mathbf{W} \cdot \mathbf{p}, \quad (1.38)$$

which defines a velocity  $\mathbf{W}$ , or at least its component parallel to  $\mathbf{p}$ , which we shall term the phase velocity of the field. Combining the two relations between the total energy and momentum, (1.28) and (1.38), we obtain

$$\mathbf{W} \cdot \mathbf{V} = c^2, \quad (1.39)$$

which implies that the magnitude of the phase velocity is never less than the speed of light.

A further conservation theorem, which is to be identified as that for angular momentum, can be deduced from the linear momentum conservation theorem. Multiplying the  $j$ th component of (1.34) by  $x_i$  and subtracting a similar equation with  $i$  and  $j$  interchanged, we obtain

$$\frac{\partial}{\partial t}(x_i G_j - x_j G_i) = \sum_k \frac{\partial}{\partial x_k} (T_{ki} x_j - T_{kj} x_i) + T_{ij} - T_{ji}, \quad (1.40)$$

However, the stress dyadic is symmetrical,

$$T_{ij} = \delta_{ij} \frac{e^2 + b^2}{2} - e_i e_j - b_i b_j = T_{ji}, \quad (1.41)$$

and therefore (in vector notation)

$$\frac{\partial}{\partial t}(\mathbf{r} \times \mathbf{G}) + \nabla \cdot (-\mathbb{T} \times \mathbf{r}) = \mathbf{0}, \quad (1.42)$$

which implies that the total angular momentum

$$\mathbf{L} = \int (\mathrm{d}\mathbf{r}) \mathbf{r} \times \mathbf{G} \quad (1.43)$$

of a field confined to a finite spatial volume is constant in time.

In the presence of electric charge, the energy and momentum of the electromagnetic field are no longer conserved. It is easily shown that

$$\frac{\partial}{\partial t} U + \nabla \cdot \mathbf{S} = -\mathbf{j} \cdot \mathbf{e}, \quad (1.44a)$$

$$\frac{\partial}{\partial t} \mathbf{G} + \nabla \cdot \mathbb{T} = -(\rho \mathbf{e} + \frac{1}{c} \mathbf{j} \times \mathbf{b}), \quad (1.44b)$$

implying that electromagnetic energy is destroyed at the rate of  $\mathbf{j} \cdot \mathbf{e}$  per unit volume, and that  $\rho \mathbf{e} + \frac{1}{c} \mathbf{j} \times \mathbf{b}$  measures the rate of annihilation of linear electromagnetic momentum, per unit volume. In a region that includes only the  $a$ th elementary charge, electromagnetic energy and momentum disappear at a rate  $q_a \mathbf{v}_a \cdot \mathbf{e}(\mathbf{r}_a)$ , and  $q_a (\mathbf{e}(\mathbf{r}_a) + \frac{1}{c} \mathbf{v}_a \times \mathbf{b}(\mathbf{r}_a))$ , respectively. If the indestructibility of energy and momentum is to be preserved, these expressions must equal the rate of increase of the energy and linear momentum of the  $a$ th elementary charge,

$$\frac{\mathrm{d}E_a}{\mathrm{d}t} = q_a \mathbf{v}_a \cdot \mathbf{e}(\mathbf{r}_a), \quad (1.45a)$$

$$\frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t} = q_a \left( \mathbf{e}(\mathbf{r}_a) + \frac{1}{c} \mathbf{v}_a \times \mathbf{b}(\mathbf{r}_a) \right) = \mathbf{F}_a, \quad (1.45b)$$

which determines the force,  $\mathbf{F}_a$ , exerted on the  $a$ th charge by the electromagnetic field, in terms of the rate of change of mechanical momentum  $\mathbf{p}_a = m_a \mathbf{v}_a$ . The consistency of the definitions adopted for field energy and momentum is

verified by the observation that the rate of increase of the energy of the  $a$ th particle, in accord with mechanical principles, is equal to the rate at which the force  $\mathbf{F}_a$  does work on the particle,

$$\frac{dE_a}{dt} = \mathbf{F}_a \cdot \mathbf{v}_a . \quad (1.46)$$

In a similar fashion, the rate of loss of electromagnetic angular momentum per unit volume  $\mathbf{r} \times (\rho \mathbf{e} + \frac{1}{c} \mathbf{j} \times \mathbf{b})$ , when integrated over a region enclosing the  $a$ th charge, must equal the rate of increase of  $\mathbf{L}_a$ , the angular momentum of the particle,

$$\frac{d\mathbf{L}_a}{dt} = q_a \mathbf{r}_a \times \left( \mathbf{e}(\mathbf{r}_a) + \frac{1}{c} \mathbf{v}_a \times \mathbf{b}(\mathbf{r}_a) \right) = \mathbf{r}_a \times \mathbf{F}_a . \quad (1.47)$$

The identification of electromagnetic angular momentum is confirmed by this result, that the rate at which the angular momentum of the particle increases equals the moment of the force acting on it. For a further discussion of the local conservation of energy and momentum see Problem 1.31.

## 1.2 Variational Principle

The equations of motion of the field and matter can be expressed in the compact form of a variational principle or Hamilton's principle. It is first convenient to introduce suitable coordinates for the field. These we shall choose as the vector potential  $\mathbf{a}$  and the scalar potential  $\phi$ , defined by

$$\mathbf{e} = -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{a} - \nabla \phi , \quad \mathbf{b} = \nabla \times \mathbf{a} , \quad (1.48)$$

which ensures that the second set of field equations (1.19b) is satisfied identically. The potentials are not uniquely determined by these equations; rather, the set of potentials

$$\mathbf{a}' = \mathbf{a} - \nabla \psi , \quad \phi' = \phi + \frac{1}{c} \frac{\partial}{\partial t} \psi \quad (1.49)$$

leads to the same field intensities as  $\mathbf{a}$  and  $\phi$ , for arbitrary  $\psi$ . Such a modification of the potentials is referred to as a gauge transformation, and those quantities which are unaltered by the transformation are called gauge invariant. The absence of a precise definition for the potentials will cause no difficulty provided that all physical quantities expressed in terms of the potentials are required to be gauge invariant.

A mechanical system is completely characterized by a Lagrangian  $L$ , which is such that  $\int_{t_0}^{t_1} dt L$  is an extremal for the actual motion of the system, in comparison with all neighboring states with prescribed values of the coordinates at times  $t_0$  and  $t_1$ ,

$$\delta \int_{t_0}^{t_1} dt L = 0 . \quad (1.50)$$

We consider a general Lagrangian for the system of fields and matter which depends upon the positions and velocities of the particles, and the potentials and field quantities descriptive of the field. From the standpoint of the field, the Lagrangian is best regarded as the volume integral of a Lagrangian density  $\mathcal{L}$ . Thus, the effect of an arbitrary variation of the vector potential is expressed by

$$\begin{aligned} \delta_{\mathbf{a}} L &= \int (d\mathbf{r}) \left( \frac{\partial \mathcal{L}}{\partial \mathbf{a}} \cdot \delta \mathbf{a} + \frac{\partial \mathcal{L}}{\partial \mathbf{b}} \cdot \nabla \times \delta \mathbf{a} - \frac{1}{c} \frac{\partial \mathcal{L}}{\partial \mathbf{e}} \cdot \delta \dot{\mathbf{a}} \right) \\ &= \int (d\mathbf{r}) \left( \frac{\delta L}{\delta \mathbf{a}} \cdot \delta \mathbf{a} + \frac{\delta L}{\delta \dot{\mathbf{a}}} \cdot \delta \dot{\mathbf{a}} \right) , \end{aligned} \quad (1.51)$$

in which we have introduced the variational derivatives,

$$\frac{\delta L}{\delta \mathbf{a}} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}} + \nabla \times \frac{\partial \mathcal{L}}{\partial \mathbf{b}} , \quad (1.52a)$$

$$\frac{\delta L}{\delta \dot{\mathbf{a}}} = -\frac{1}{c} \frac{\partial \mathcal{L}}{\partial \mathbf{e}} , \quad (1.52b)$$

and discarded a surface integral by requiring that all variations vanish on the spatial boundary of the region, as well as at the initial and terminal times  $t_0$  and  $t_1$ . In a similar fashion,

$$\delta_{\phi} L = \int (d\mathbf{r}) \left( \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi - \frac{\partial \mathcal{L}}{\partial \mathbf{e}} \cdot \nabla \delta \phi \right) = \int (d\mathbf{r}) \frac{\delta L}{\delta \phi} \delta \phi , \quad (1.53)$$

with

$$\frac{\delta L}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi} + \nabla \cdot \left( \frac{\partial \mathcal{L}}{\partial \mathbf{e}} \right) , \quad (1.54)$$

provided the time derivative of the scalar potential is absent in the Lagrangian. These relations, (1.51) and (1.53), expressed in terms of variational derivatives, are formally analogous to the variation of a Lagrangian associated with a material particle's coordinates,

$$\delta_{\mathbf{r}_a} L = \frac{\partial L}{\partial \mathbf{r}_a} \cdot \delta \mathbf{r}_a + \frac{\partial L}{\partial \mathbf{v}_a} \cdot \frac{d}{dt} \delta \mathbf{r}_a . \quad (1.55)$$

Therefore, the condition expressing the stationary character of  $\int_{t_0}^{t_1} dt L$  for variations of  $\mathbf{r}_a$ , subject to the vanishing of all variations at the termini,

$$\frac{d}{dt} \frac{\partial L}{\partial \mathbf{v}_a} = \frac{\partial L}{\partial \mathbf{r}_a} , \quad (1.56)$$

has a formally similar aspect for variations of  $\mathbf{a}$  and  $\phi$ ,

$$\frac{\partial}{\partial t} \frac{\delta L}{\delta \dot{\mathbf{a}}} = \frac{\delta L}{\delta \mathbf{a}}, \quad 0 = \frac{\delta L}{\delta \phi}. \quad (1.57)$$

Hence, the field equations deduced from a variational principle are

$$-\nabla \times \frac{\partial \mathcal{L}}{\partial \mathbf{b}} = \frac{1}{c} \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \mathbf{e}} + \frac{\partial \mathcal{L}}{\partial \mathbf{a}}, \quad \nabla \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{e}} = -\frac{\partial \mathcal{L}}{\partial \phi}, \quad (1.58)$$

which are identical with the Maxwell-Lorentz equations (1.19a) if

$$\mathcal{L} = \frac{e^2 - b^2}{2} - \rho\phi + \frac{1}{c} \mathbf{j} \cdot \mathbf{a}. \quad (1.59)$$

The Lagrangian thus consists of a part involving only the field quantities,

$$L_f = \int (\mathrm{d}\mathbf{r}) \frac{e^2 - b^2}{2}, \quad (1.60)$$

a part containing the coordinates of both field and matter,

$$L_{fm} = - \int (\mathrm{d}\mathbf{r}) \left( \rho\phi - \frac{1}{c} \mathbf{j} \cdot \mathbf{a} \right) = - \sum_a q_a \left( \phi(\mathbf{r}_a) - \frac{1}{c} \mathbf{v}_a \cdot \mathbf{a}(\mathbf{r}_a) \right), \quad (1.61)$$

and a part involving only material quantities, which, as we shall verify, is for nonrelativistic particles

$$L_m = \sum_a \frac{1}{2} m_a v_a^2. \quad (1.62)$$

(For the relativistic generalization see Problem 1.32.) The Lagrangian form of the  $a$ th particle's equation of motion (1.56) is

$$\frac{\mathrm{d}}{\mathrm{d}t} \left( m_a \mathbf{v}_a + \frac{q_a}{c} \mathbf{a}(\mathbf{r}_a) \right) = -q_a \nabla_{\mathbf{r}_a} \left( \phi(\mathbf{r}_a) - \frac{1}{c} \mathbf{v}_a \cdot \mathbf{a}(\mathbf{r}_a) \right), \quad (1.63)$$

where we see the appearance of the canonical momentum,

$$\boldsymbol{\pi}_a = m_a \mathbf{v}_a + \frac{q_a}{c} \mathbf{a}(\mathbf{r}_a). \quad (1.64)$$

However,

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{a}(\mathbf{r}_a, t) = \frac{\partial}{\partial t} \mathbf{a} + \mathbf{v}_a \cdot \nabla \mathbf{a} = \frac{\partial}{\partial t} \mathbf{a} - \mathbf{v}_a \times \mathbf{b} + \nabla(\mathbf{v}_a \cdot \mathbf{a}), \quad (1.65)$$

for in computing the time derivative, the implicit dependence of the particle's position on the time cannot be ignored. It is thus confirmed that the Lorentz force law (1.45b) holds,

$$\frac{\mathrm{d}}{\mathrm{d}t} m_a \mathbf{v}_a = q_a \left( \mathbf{e}(\mathbf{r}_a) + \frac{1}{c} \mathbf{v}_a \times \mathbf{b}(\mathbf{r}_a) \right). \quad (1.66)$$

### 1.3 Conservation Theorems

The various conservation laws, those of charge, energy, linear momentum, and angular momentum, are consequences of the invariance of Hamilton's principle under certain transformations. These are, respectively, gauge transformations, temporal displacements, spatial translations, and spatial rotations. A gauge transformation (1.49) induces the variation

$$\delta \mathbf{a} = -\nabla \psi, \quad \delta \phi = \frac{1}{c} \frac{\partial}{\partial t} \psi, \quad (1.67a)$$

$$\delta \mathbf{e} = \delta \mathbf{b} = \mathbf{0}, \quad (1.67b)$$

whence

$$\delta L = \int (\mathrm{d}\mathbf{r}) \left( \nabla \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{a}} - \frac{1}{c} \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \phi} \right) \psi + \frac{1}{c} \frac{\mathrm{d}}{\mathrm{d}t} \int (\mathrm{d}\mathbf{r}) \frac{\partial \mathcal{L}}{\partial \phi} \psi, \quad (1.68)$$

from which we can infer from (1.59) that the local charge conservation equation,

$$\nabla \cdot \left( c \frac{\partial \mathcal{L}}{\partial \mathbf{a}} \right) + \frac{\partial}{\partial t} \left( -\frac{\partial \mathcal{L}}{\partial \phi} \right) = \nabla \cdot \mathbf{j} + \frac{\partial}{\partial t} \rho = 0, \quad (1.69)$$

must be a consequence of the field equations, for  $\int_{t_0}^{t_1} \mathrm{d}t L$  is stationary with respect to arbitrary independent variations of  $\mathbf{a}$  and  $\phi$ .

The value of  $\int_{t_0}^{t_1} \mathrm{d}t L$  is in no way affected by an alteration of the time origin,

$$\int_{t_0 - \delta t}^{t_1 - \delta t} \mathrm{d}t L(t + \delta t) - \int_{t_0}^{t_1} \mathrm{d}t L(t) = 0, \quad (1.70)$$

where  $\delta t$  is an arbitrary constant. We may conceive of the time displacement as a variation of the system's coordinates which consists in replacing the actual values at time  $t$  by the actual values which the system will assume at time  $t + \delta t$ . The statement of invariance with respect to the origin of time now reads

$$\int_{t_0}^{t_1} \mathrm{d}t \delta L = \delta t (L(t_1) - L(t_0)) = \delta t \int_{t_0}^{t_1} \mathrm{d}t \frac{\mathrm{d}L}{\mathrm{d}t}, \quad (1.71)$$

where  $\delta L$  is the consequence of the variations

$$\delta \mathbf{a} = \delta t \dot{\mathbf{a}}, \quad \delta \phi = \delta t \dot{\phi}, \quad \delta \mathbf{r}_a = \delta t \mathbf{v}_a, \quad (1.72a)$$

$$\delta L = \delta t \left[ \int (\mathrm{d}\mathbf{r}) \left( \frac{\delta L}{\delta \mathbf{a}} \cdot \dot{\mathbf{a}} + \frac{\delta L}{\delta \dot{\mathbf{a}}} \cdot \ddot{\mathbf{a}} + \frac{\delta L}{\delta \phi} \cdot \dot{\phi} \right) + \sum_a \left( \frac{\partial L}{\partial \mathbf{r}_a} \cdot \mathbf{v}_a + \frac{\partial L}{\partial \mathbf{v}_a} \cdot \dot{\mathbf{v}}_a \right) \right]. \quad (1.72b)$$

In writing this expression for  $\delta L$  various surface integrals have been discarded. This can no longer be justified by the statement that the variation vanishes at

the surface of the integration region, for it is not possible to satisfy this condition with the limited type of variation that is being contemplated. Rather, it is assumed for simplicity that the volume integration encompasses the entire field. On rearranging the terms of  $\delta L$  and employing the Lagrangian equations of motion (1.56), (1.57), we obtain

$$\delta L = \delta t \frac{d}{dt} \left( \int (d\mathbf{r}) \frac{\delta L}{\delta \dot{\mathbf{a}}} \cdot \dot{\mathbf{a}} + \sum_a \frac{\partial L}{\partial \mathbf{v}_a} \cdot \mathbf{v}_a \right), \quad (1.73)$$

from which it follows from (1.71) that

$$E = \int (d\mathbf{r}) \frac{\delta L}{\delta \dot{\mathbf{a}}} \cdot \dot{\mathbf{a}} + \sum_a \frac{\partial L}{\partial \mathbf{v}_a} \cdot \mathbf{v}_a - L \quad (1.74)$$

is independent of time. It is easily verified from (1.59) that  $E$  is the total energy of the system,

$$E = \int (d\mathbf{r}) \frac{e^2 + b^2}{2} + \sum_a \frac{1}{2} m_a v_a^2. \quad (1.75)$$

The Lagrangian is unaltered by an arbitrary translation of the position variable of integration, that is, if  $\mathbf{r}$  is replaced by  $\mathbf{r} + \delta\mathbf{r}$ , with  $\delta\mathbf{r}$  an arbitrary constant vector. The region of integration must be suitably modified, of course, but this need not be considered if the entire field is included, for the limits of integration are then effectively infinite. Under this substitution, the matter part of the Lagrangian, which corresponds to the Lagrange density  $\mathcal{L}_m(\mathbf{r}) = L_m(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}_a)$ , is replaced by  $L_m(\mathbf{r} + \delta\mathbf{r}) \delta(\mathbf{r} + \delta\mathbf{r} - \mathbf{r}_a)$ . Hence, viewed as the variation

$$\delta \mathbf{a} = (\delta \mathbf{r} \cdot \nabla) \mathbf{a}, \quad \delta \phi = (\delta \mathbf{r} \cdot \nabla) \phi, \quad \delta \mathbf{r}_a = -\delta \mathbf{r}, \quad (1.76)$$

the translation of the space coordinate system induces a variation of

$$\begin{aligned} \delta L = \int (d\mathbf{r}) & \left[ \frac{\delta L}{\delta \mathbf{a}} \cdot (\delta \mathbf{r} \cdot \nabla) \mathbf{a} + \frac{\delta L}{\delta \dot{\mathbf{a}}} \cdot (\delta \mathbf{r} \cdot \nabla) \dot{\mathbf{a}} + \frac{\delta L}{\delta \phi} (\delta \mathbf{r} \cdot \nabla) \phi \right] \\ & - \sum_a \frac{\partial L}{\partial \mathbf{r}_a} \cdot \delta \mathbf{r}, \end{aligned} \quad (1.77)$$

which must be zero. As a consequence of the Lagrangian equations of motion (1.56), (1.57) and the relations

$$(\delta \mathbf{r} \cdot \nabla) \mathbf{a} = \nabla (\delta \mathbf{r} \cdot \mathbf{a}) + \mathbf{b} \times \delta \mathbf{r}, \quad (1.78a)$$

$$\nabla \cdot \left( \frac{\delta L}{\delta \dot{\mathbf{a}}} \right) = -\frac{1}{c} \nabla \cdot \left( \frac{\partial \mathcal{L}}{\partial \mathbf{e}} \right) = -\frac{1}{c} \rho, \quad (1.78b)$$

we obtain

$$\delta L = -\frac{d}{dt} \left[ -\int (d\mathbf{r}) \frac{\delta L}{\delta \dot{\mathbf{a}}} \times \mathbf{b} + \sum_a \left( \frac{\partial L}{\partial \mathbf{v}_a} - \frac{1}{c} q_a \mathbf{a}(\mathbf{r}_a) \right) \right] \cdot \delta \mathbf{r} = 0. \quad (1.79)$$

Therefore,

$$\begin{aligned} \mathbf{P} &= -\int (d\mathbf{r}) \frac{\delta L}{\delta \dot{\mathbf{a}}} \times \mathbf{b} + \sum_a \left( \frac{\partial L}{\partial \mathbf{v}_a} - \frac{1}{c} q_a \mathbf{a}(\mathbf{r}_a) \right) \\ &= \int (d\mathbf{r}) \frac{1}{c} \mathbf{e} \times \mathbf{b} + \sum_a m_a \mathbf{v}_a, \end{aligned} \quad (1.80)$$

the total linear momentum of the system, must be constant in time.

Similar considerations are applicable to a rotation of the coordinate system. The infinitesimal rotation

$$\mathbf{r} \rightarrow \mathbf{r} + \boldsymbol{\epsilon} \times \mathbf{r} \quad (1.81)$$

induces the variation (because  $\mathbf{a}$ , like  $\mathbf{r}_a$ , is a vector)

$$\delta \mathbf{a} = (\boldsymbol{\epsilon} \times \mathbf{r} \cdot \nabla) \mathbf{a} - \boldsymbol{\epsilon} \times \mathbf{a}, \quad \delta \phi = (\boldsymbol{\epsilon} \times \mathbf{r} \cdot \nabla) \phi, \quad \delta \mathbf{r}_a = -\boldsymbol{\epsilon} \times \mathbf{r}_a, \quad (1.82)$$

which must leave the Lagrangian unaltered,

$$\begin{aligned} \delta L &= \int (d\mathbf{r}) \left\{ \frac{\delta L}{\delta \mathbf{a}} \cdot [(\boldsymbol{\epsilon} \cdot \mathbf{r} \times \nabla) \mathbf{a} - \boldsymbol{\epsilon} \times \mathbf{a}] + \frac{\delta L}{\delta \dot{\mathbf{a}}} \cdot [(\boldsymbol{\epsilon} \cdot \mathbf{r} \times \nabla) \dot{\mathbf{a}} - \boldsymbol{\epsilon} \times \dot{\mathbf{a}}] \right. \\ &\quad \left. + \frac{\delta L}{\delta \phi} (\boldsymbol{\epsilon} \cdot \mathbf{r} \times \nabla) \phi \right\} - \sum_a \boldsymbol{\epsilon} \cdot \mathbf{r}_a \times \frac{\partial L}{\partial \mathbf{r}_a} - \sum_a \boldsymbol{\epsilon} \cdot \mathbf{v}_a \times \frac{\partial L}{\partial \mathbf{v}_a} = 0. \end{aligned} \quad (1.83)$$

However, again using (1.78b),

$$\delta L = -\frac{d}{dt} \left[ -\int (d\mathbf{r}) \mathbf{r} \times \left( \frac{\delta L}{\delta \dot{\mathbf{a}}} \times \mathbf{b} \right) + \sum_a \mathbf{r}_a \times \left( \frac{\partial L}{\partial \mathbf{v}_a} - \frac{1}{c} q_a \mathbf{a}(\mathbf{r}_a) \right) \right] \cdot \boldsymbol{\epsilon}, \quad (1.84)$$

in consequence of the identity

$$(\boldsymbol{\epsilon} \cdot \mathbf{r} \times \nabla) \mathbf{a} - \boldsymbol{\epsilon} \times \mathbf{a} = \nabla(\boldsymbol{\epsilon} \cdot \mathbf{r} \times \mathbf{a}) + \mathbf{b} \times (\boldsymbol{\epsilon} \times \mathbf{r}). \quad (1.85)$$

Therefore,

$$\mathbf{L} = \int (d\mathbf{r}) \mathbf{r} \times \left( \frac{1}{c} \mathbf{e} \times \mathbf{b} \right) + \sum_a m_a \mathbf{r}_a \times \mathbf{v}_a, \quad (1.86)$$

the total angular momentum, is unchanged in time.

## 1.4 Delta Function

Preparatory to determining the fields produced by given distributions of charge and current, it is useful to consider some properties of the  $\delta$  function, and in particular, its connections with the Fourier integral theorem. A one-dimensional  $\delta$  function is defined by the statements

$$\int_{x_0}^{x_1} dx \delta(x) = \begin{cases} 1, & x_1 > 0 > x_0, \\ 0, & x_1 > x_0 > 0, \text{ or } 0 > x_1 > x_0, \end{cases} \quad (1.87)$$

that is, the integral vanishes unless the domain of integration includes the origin, when the value assumed by the integral is unity. The function  $\delta(x - x')$  has corresponding properties relative to the point  $x'$ . Particular examples of functions possessing these attributes in the limit are

$$\delta(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}, \quad (1.88a)$$

$$\delta(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} e^{-\pi x^2 / \epsilon^2}. \quad (1.88b)$$

An integral representation for  $\delta(x)$  can be constructed from the formulæ

$$\frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} e^{-\epsilon|k|}, \quad (1.89a)$$

$$\frac{1}{\epsilon} e^{-\pi x^2 / \epsilon^2} = \frac{1}{2\pi} \int dk e^{ikx} e^{-\epsilon^2 k^2 / 4\pi}. \quad (1.89b)$$

If we perform the limiting operation under the integral sign, either expression yields

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} = \frac{1}{\pi} \int_0^{\infty} dk \cos kx. \quad (1.90)$$

The three-dimensional  $\delta$  function already introduced (1.5) is correctly represented by

$$\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z), \quad (1.91)$$

for  $\delta(\mathbf{r})$  certainly vanishes unless  $x$ ,  $y$ , and  $z$  are simultaneously zero, and the integral over any volume enclosing the origin is unity. More generally,

$$\delta(\mathbf{r} - \mathbf{r}') = \delta(x - x')\delta(y - y')\delta(z - z'). \quad (1.92)$$

The representation for  $\delta(\mathbf{r})$ , obtained by multiplying individual integrals (1.90) for the one-dimensional delta functions can be regarded as an integral extended over the entirety of the space associated with the vector  $\mathbf{k}$ ,

$$\delta(\mathbf{r}) = \frac{1}{(2\pi)^3} \int (d\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}. \quad (1.93)$$

The functional representations mentioned previously, (1.7a), (1.7b), are consequences of the formulæ

$$\frac{1}{\pi^2} \frac{\epsilon}{(r^2 + \epsilon^2)^2} = \frac{1}{(2\pi)^3} \int (d\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} e^{-\epsilon k}, \quad (1.94a)$$

$$\frac{1}{\epsilon^3} e^{-\pi r^2/\epsilon^2} = \frac{1}{(2\pi)^3} \int (d\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} e^{-\epsilon^2 k^2/4\pi}. \quad (1.94b)$$

An arbitrary function of a coordinate  $x$  can be represented by a linear superposition of  $\delta$  functions,

$$f(x) = \int_{-\infty}^{\infty} dx' \delta(x - x') f(x'), \quad (1.95)$$

for the entire contribution to the integral comes from the point  $x' = x$ . On employing the integral representation (1.90) for  $\delta(x - x')$ , we obtain

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \int_{-\infty}^{\infty} dx' e^{-ikx'} f(x'), \quad (1.96)$$

which states the possibility of constructing an arbitrary function from the elementary periodic function  $e^{ikx}$  – the Fourier integral theorem. The corresponding statements in three dimensions are

$$\begin{aligned} f(\mathbf{r}) &= \int (d\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') f(\mathbf{r}') \\ &= \frac{1}{(2\pi)^3} \int (d\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} \int (d\mathbf{r}') e^{-i\mathbf{k}\cdot\mathbf{r}'} f(\mathbf{r}'), \end{aligned} \quad (1.97)$$

while a function of space and time is represented by

$$\begin{aligned} f(\mathbf{r}, t) &= \int (d\mathbf{r}') dt' \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') f(\mathbf{r}', t') \\ &= \frac{1}{(2\pi)^4} \int (d\mathbf{k}) d\omega e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \int (d\mathbf{r}') dt' e^{-i(\mathbf{k}\cdot\mathbf{r}' - \omega t')} f(\mathbf{r}', t'). \end{aligned} \quad (1.98)$$

Thus, an arbitrary function  $f(\mathbf{r}, t)$  can be synthesized by a proper superposition of the functions  $\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$ , which are the mathematical descriptions of plane waves, harmonic disturbances propagating in the direction of the vector  $\mathbf{k}$ , with a space periodicity length or wavelength  $\lambda = 2\pi/|\mathbf{k}|$ , and a time periodicity or period  $T = 2\pi/\omega$ .

## 1.5 Radiation Fields

The treatment of an electrodynamic problem involves two preliminary stages; the evaluation of the fields produced by a given array of charges moving in a prescribed fashion, and the determination of the motion of a charge acted on by a given electromagnetic field. The correct solution of the problem is obtained when these two aspects of the situation are consistent, that is, when

the charges move in such a way that the fields they generate produce precisely this state of motion. We turn to a discussion of the first stage, the calculation of the fields produced by a given distribution of charge and current.

The auxiliary quantities, the vector and scalar potentials, have been introduced in order to satisfy identically the second set of field equations (1.19b). Determining equations for the potentials are obtained on substituting the representations (1.48) for  $\mathbf{e}$  and  $\mathbf{b}$  in the first set of equations (1.19a), with the result

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \phi = -\frac{1}{c} \frac{\partial}{\partial t} \left(\nabla \cdot \mathbf{a} + \frac{1}{c} \frac{\partial \phi}{\partial t}\right) - \rho, \quad (1.99a)$$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \mathbf{a} = \nabla \left(\nabla \cdot \mathbf{a} + \frac{1}{c} \frac{\partial \phi}{\partial t}\right) - \frac{1}{c} \mathbf{j}. \quad (1.99b)$$

It is always possible to impose the condition

$$\nabla \cdot \mathbf{a} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0, \quad (1.100)$$

for if this quantity does not vanish, one can, by a suitable gauge transformation, introduce new potentials for which the condition is valid. Thus if  $\mathbf{a}'$ ,  $\phi'$  are obtained from  $\mathbf{a}$  and  $\phi$  by a gauge transformation associated with the function  $\psi$ , as in (1.49)

$$\nabla \cdot \mathbf{a}' + \frac{1}{c} \frac{\partial \phi'}{\partial t} = \nabla \cdot \mathbf{a} + \frac{1}{c} \frac{\partial \phi}{\partial t} - \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \psi, \quad (1.101)$$

and  $\psi$  can always be chosen to produce the desired result. With this restriction upon the potentials, which is referred to as the Lorentz condition, the determining equations for the potentials become

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \phi = -\rho, \quad (1.102a)$$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \mathbf{a} = -\frac{1}{c} \mathbf{j}. \quad (1.102b)$$

It should be noted that the potentials are still not unique, for a gauge transformation, with the scalar function  $\psi$  satisfying

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \psi = 0 \quad (1.103)$$

is compatible with the Lorentz condition.

The charge and current densities, as prescribed functions of the space and time coordinates, can be represented in terms of plane waves as in (1.98). Thus,

$$\rho(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \int (d\mathbf{r}') dt' \int (d\mathbf{k}) d\omega e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}') - i\omega(t - t')} \rho(\mathbf{r}', t'). \quad (1.104)$$

The advantage of this Fourier integral representation is that a particular solution for the potentials can be constructed by inspection. For example, from (1.102a),

$$\phi(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \int (d\mathbf{k}) d\omega \int (d\mathbf{r}') dt' \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-i\omega(t-t')}}{k^2 - \omega^2/c^2} \rho(\mathbf{r}', t'). \quad (1.105)$$

As a first step in the simplification of this result, consider the Green's function

$$G(\mathbf{r}) = \int \frac{(d\mathbf{k})}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2 - \omega^2/c^2}, \quad (1.106)$$

which is a solution of the differential equation

$$\left( \nabla^2 + \frac{\omega^2}{c^2} \right) G(\mathbf{r}) = -\delta(\mathbf{r}). \quad (1.107)$$

Upon introducing polar coordinates in the  $\mathbf{k}$  space, we obtain

$$G(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d\theta \sin \theta \int 2\pi k^2 dk \frac{e^{ikr \cos \theta}}{k^2 - \omega^2/c^2} = \frac{1}{2\pi^2 r} \int_0^\infty k dk \frac{\sin kr}{k^2 - \omega^2/c^2}, \quad (1.108)$$

or, equivalently,

$$G(\mathbf{r}) = -\frac{i}{8\pi^2 r} \int_{-\infty}^\infty dk \left( \frac{e^{ikr}}{k - \omega/c} + \frac{e^{ikr}}{k + \omega/c} \right). \quad (1.109)$$

An essential complication can no longer be ignored; the integrand becomes infinite at  $k = \pm\omega/c$ . The difficulty can be avoided in a purely formal manner by supposing that  $1/c$  has a small imaginary part which will be eventually be allowed to vanish. If the imaginary part of  $1/c$  is positive,<sup>2</sup> the integrand, considered as a function of the complex variable  $k$ , has a simple pole at  $\omega/c$  in the upper half-plane, and a simple pole at  $-\omega/c$  in the lower half-plane. The path of integration along the real axis can be closed by an infinite semicircle drawn in the upper half-plane without affecting the value of the integral, since  $r$  is positive. Within this closed contour the integrand is everywhere analytic save at the simple pole at  $k = \omega/c$ . Hence, by the theorem of residues,

$$G(\mathbf{r}) = \frac{e^{i\omega r/c}}{4\pi r}. \quad (1.110)$$

If the imaginary part of  $1/c$  is negative, the position of the poles are reflected in the real axis, and the pole at  $k = -\omega/c$  lies in the upper half-plane. For this situation,

$$G(\mathbf{r}) = \frac{e^{-i\omega r/c}}{4\pi r}. \quad (1.111)$$

<sup>2</sup> This is equivalent to distorting the  $k$  contour to avoid the poles by passing below the pole at  $+\omega/c$ , and above the pole at  $-\omega/c$ .

It can be directly verified that the two functions  $e^{\pm i\omega r/c}/(4\pi r)$  are solutions of the differential equation (1.107) for  $G(\mathbf{r})$ . It must be shown that  $(\nabla^2 + \omega^2/c^2)e^{\pm i\omega r/c}/(4\pi r)$  has the properties of  $-\delta(\mathbf{r})$ , which will be achieved on demonstrating that

$$\int (d\mathbf{r}) \left( \nabla^2 + \frac{\omega^2}{c^2} \right) \frac{e^{\pm i\omega r/c}}{4\pi r} = -1 \quad (1.112)$$

for any region of integration that includes the origin. It is sufficient to consider a sphere of arbitrary radius  $R$ . Thus, we are required to prove that

$$R^2 \frac{d}{dR} \left( \frac{e^{\pm i\omega R/c}}{R} \right) + \frac{\omega^2}{c^2} \int_0^R r dr e^{\pm i\omega r/c} = -1, \quad (1.113)$$

which is easily checked. It is apparent, then, that the difficulty encountered by the Fourier integral method arises from the existence of two solutions for  $G(\mathbf{r})$  and, in consequence, for the potentials. Which of these solutions to adopt can only be decided by additional physical considerations.

Tentatively choosing (1.110), we obtain from (1.105)

$$\phi(\mathbf{r}, t) = \frac{1}{2\pi} \int d\omega (d\mathbf{r}') dt' \frac{e^{i\omega|\mathbf{r}-\mathbf{r}'|/c - i\omega(t-t')}}{4\pi|\mathbf{r}-\mathbf{r}'|} \rho(\mathbf{r}', t'). \quad (1.114)$$

The integral with respect to  $\omega$  is recognized as that of a delta function,

$$\phi(\mathbf{r}, t) = \int (d\mathbf{r}') dt' \frac{\delta(t' - t + |\mathbf{r} - \mathbf{r}'|/c)}{4\pi|\mathbf{r} - \mathbf{r}'|} \rho(\mathbf{r}', t'), \quad (1.115)$$

and if the integration with respect to  $t'$  is performed,

$$\phi(\mathbf{r}, t) = \int (d\mathbf{r}') \frac{\rho(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{4\pi|\mathbf{r} - \mathbf{r}'|}. \quad (1.116)$$

This result expresses the scalar potential at the point  $\mathbf{r}$  and time  $t$  in terms of the charge density at other points of space and earlier times, the time interval being just that required to traverse the spatial separation at the speed  $c$ . The formula thus contains a concise description of the propagation of electromagnetic fields at the speed of light. Evidently, had the solution  $e^{-i\omega r/c}/(4\pi r)$  been adopted for  $G(\mathbf{r})$ , the evaluation of the potential at a time  $t$  would have involved a knowledge of the charge density at later times. This possibility must be rejected, for it requires information which, by the nature of the physical world, is unavailable.<sup>3</sup> The corresponding solution of (1.102b) for the vector potential, in its several stages of development, is

<sup>3</sup> However, it is actually possible to use advanced Green's functions, with suitable boundary conditions, to describe classical physics. See [11]. This led Feynman to the discovery of the causal or Feynman propagator. See Problem 1.37.

$$\begin{aligned}
\mathbf{a}(\mathbf{r}, t) &= \frac{1}{2\pi} \int d\omega \int (d\mathbf{r}') dt' \frac{e^{i\omega|\mathbf{r}-\mathbf{r}'|/c - i\omega(t-t')}}{4\pi|\mathbf{r}-\mathbf{r}'|} \frac{1}{c} \mathbf{j}(\mathbf{r}', t') \\
&= \int (d\mathbf{r}') dt' \frac{\delta(t' - t + |\mathbf{r} - \mathbf{r}'|/c)}{4\pi|\mathbf{r} - \mathbf{r}'|} \frac{1}{c} \mathbf{j}(\mathbf{r}', t') \\
&= \frac{1}{c} \int (d\mathbf{r}') \frac{\mathbf{j}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{4\pi|\mathbf{r} - \mathbf{r}'|} . \tag{1.117}
\end{aligned}$$

These solutions for the vector and scalar potentials, the so-called retarded potentials, satisfy the Lorentz condition. This is most easily demonstrated with the form the potentials assume before the integration with respect to the time  $t'$ . The quantity  $\delta(t' - t + |\mathbf{r} - \mathbf{r}'|/c)/(4\pi|\mathbf{r} - \mathbf{r}'|)$  involves only the difference of time and space coordinates. Therefore derivatives with respect to  $t$  or  $\mathbf{r}$  can be replaced by corresponding derivatives acting on  $t'$  and  $\mathbf{r}'$ , with a compensating sign change. Hence, with a suitable integration by parts,

$$\begin{aligned}
\nabla \cdot \mathbf{a}(\mathbf{r}, t) + \frac{1}{c} \frac{\partial}{\partial t} \phi(\mathbf{r}, t) &= \frac{1}{c} \int (d\mathbf{r}') dt' \frac{\delta(t' - t + |\mathbf{r} - \mathbf{r}'|/c)}{4\pi|\mathbf{r} - \mathbf{r}'|} \\
&\quad \times \left( \nabla' \cdot \mathbf{j}(\mathbf{r}', t') + \frac{\partial}{\partial t'} \rho(\mathbf{r}', t') \right) = 0 , \tag{1.118}
\end{aligned}$$

in consequence of the conservation of charge, (1.14).

As a particular example, consider a point charge moving in a prescribed fashion, that is, its position  $\mathbf{r}(t)$  and velocity  $\mathbf{v}(t)$  are given functions of time. The charge and current densities are, accordingly, represented by

$$\rho(\mathbf{r}, t) = q \delta(\mathbf{r} - \mathbf{r}(t)) , \quad \mathbf{j}(\mathbf{r}, t) = q \mathbf{v}(t) \delta(\mathbf{r} - \mathbf{r}(t)) . \tag{1.119}$$

The most convenient form for the potentials is, again, that involving the delta function. On integrating over the space variable  $\mathbf{r}'$ , we obtain from (1.115) and (1.117)

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi} \int dt' \frac{\delta(t' - t + |\mathbf{r} - \mathbf{r}(t')|/c)}{|\mathbf{r} - \mathbf{r}(t')|} , \tag{1.120a}$$

$$\mathbf{a}(\mathbf{r}, t) = \frac{q}{4\pi} \int dt' \frac{\mathbf{v}(t')}{c} \frac{\delta(t' - t + |\mathbf{r} - \mathbf{r}(t')|/c)}{|\mathbf{r} - \mathbf{r}(t')|} . \tag{1.120b}$$

The entire contribution to these integrals comes from the time  $\tau$  defined by

$$t - \tau = \frac{|\mathbf{r} - \mathbf{r}(\tau)|}{c} , \tag{1.121}$$

which is evidently the time at which an electromagnetic field, moving at the speed  $c$ , must leave the position of the charge in order to reach the point of observation  $\mathbf{r}$  at the time  $t$ . In performing the final integration with respect to  $t'$ , one must be careful to observe that  $dt'$  is not the differential of the  $\delta$  function's argument, and that therefore a change of variable is required. Thus calling

$$\eta(t') = t' - t + \frac{|\mathbf{r} - \mathbf{r}(t')|}{c}, \quad (1.122)$$

we obtain for the scalar potential

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi} \left( \frac{\frac{d\eta'}{d\eta}}{|\mathbf{r} - \mathbf{r}(t')|} \right)_{t'=\tau}. \quad (1.123)$$

However,

$$\frac{d\eta}{dt'} = 1 - \frac{\mathbf{v}(t') \cdot \mathbf{r} - \mathbf{r}(t')}{|\mathbf{r} - \mathbf{r}(t')|}, \quad (1.124)$$

and therefore

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}(\tau)| - \mathbf{v}(\tau) \cdot (\mathbf{r} - \mathbf{r}(\tau))/c}. \quad (1.125)$$

Similarly,

$$\mathbf{a}(\mathbf{r}, t) = \frac{q}{4\pi} \frac{\frac{1}{c}\mathbf{v}(\tau)}{|\mathbf{r} - \mathbf{r}(\tau)| - \mathbf{v}(\tau) \cdot (\mathbf{r} - \mathbf{r}(\tau))/c} = \frac{\mathbf{v}(\tau)}{c} \phi(\mathbf{r}, t). \quad (1.126)$$

The direct evaluation of the fields from these potentials, the so-called Liénard-Wiechert potentials, is rather involved, for the retarded time  $\tau$  is an implicit function of  $\mathbf{r}$  and  $t$ . The calculation proceeds more easily by first deriving the fields from the  $\delta$  function representation of the potentials and then performing the integration with respect to  $t'$ . However, no details will be given. (See Problems 1.7, 1.8.)

### 1.5.1 Multipole Radiation

A problem of greater interest is that of a distribution of charge with a spatial extension sufficiently small so that the charge distribution changes only slightly in the time required for light to traverse it. Otherwise expressed, the largest frequency  $\nu = \omega/2\pi$  that occurs in the time Fourier decomposition of the charge density must be such that  $\nu a/c \ll 1$ , where  $a$  is a length, representative of the system's linear dimensions. Equivalently, the corresponding wavelength  $\lambda = c/\nu$  must be large in comparison with  $a$ . Molecular systems ( $a \sim 10^{-8}$  cm) possess this property for optical and even for ultraviolet frequencies ( $\lambda \sim 10^{-6}$  cm), and the condition  $\lambda \gg a$  is more than adequately fulfilled for wavelengths in the microwave region ( $\lambda \sim 1$  cm). Under these conditions, the difference in retarded time  $t - |\mathbf{r} - \mathbf{r}'|/c$  between various parts of a molecule is of secondary importance, and to a first approximation all retarded times can be identified with that of some fixed point in the molecule, which we shall choose as the origin of coordinates. In a more precise treatment, the difference between  $t - |\mathbf{r} - \mathbf{r}'|/c$  and  $t - r/c$  can be taken into account by expansion of the charge and current densities, as follows:

$$\rho(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c) = \rho(r', t - r/c) + (r - |\mathbf{r} - \mathbf{r}'|) \frac{1}{c} \frac{\partial}{\partial t} \rho(r', t - r/c) + \dots \quad (1.127)$$

However, we shall be concerned primarily with the field at a great distance from the center of the molecule (considered to be at rest). Rather than introduce our approximation in two steps, we proceed more directly by regarding  $r'$  as small in comparison with  $r$  wherever it occurs in the retarded potential expressions. It must not be forgotten that two approximations are thereby introduced,  $r \gg a$  and  $\lambda \gg a$ . With these remarks, we insert the Taylor series expansion

$$\frac{\rho(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} = \left( 1 - \mathbf{r}' \cdot \nabla + \frac{1}{2} (\mathbf{r}' \cdot \nabla)^2 - \dots \right) \frac{\rho(r', t - r/c)}{r} \quad (1.128)$$

in the retarded scalar potential integral (1.116)

$$\begin{aligned} 4\pi\phi(\mathbf{r}, t) = & \frac{\int(d\mathbf{r}') \rho(r', t - r/c)}{r} - \nabla \cdot \frac{\int(d\mathbf{r}') \mathbf{r}' \rho(r', t - r/c)}{r} \\ & + \frac{1}{2} \nabla \nabla : \frac{\int(d\mathbf{r}') \mathbf{r}' \mathbf{r}' \rho(r', t - r/c)}{r} - \dots \quad (1.129) \end{aligned}$$

In terms of the total charge, electric dipole moment, and electric quadrupole moment dyadic,<sup>4</sup>

$$q = \int(d\mathbf{r}) \rho(\mathbf{r}, t), \quad (1.131a)$$

$$\mathbf{d}(t) = \int(d\mathbf{r}) \mathbf{r} \rho(\mathbf{r}, t), \quad (1.131b)$$

$$\mathbf{Q}(t) = \int(d\mathbf{r}) \mathbf{r} \mathbf{r} \rho(\mathbf{r}, t), \quad (1.131c)$$

the first three terms of the expansion are

$$4\pi\phi(\mathbf{r}, t) = \frac{q}{r} - \nabla \cdot \frac{\mathbf{d}(t - r/c)}{r} + \frac{1}{2} \nabla \nabla : \frac{\mathbf{Q}(t - r/c)}{r}. \quad (1.132)$$

The notation  $\mathbf{A}:\mathbf{B}$  for dyadics designates the scalar product,

$$\mathbf{A}:\mathbf{B} = \sum_{i,j} A_{ij} B_{ji}. \quad (1.133)$$

In a similar fashion, the expansion of the vector potential (1.117) is

<sup>4</sup> Usually, the electric quadrupole dyadic is defined by

$$\mathbf{Q} = 3 \int(d\mathbf{r}) \left( \mathbf{r} \mathbf{r} - \frac{1}{3} r^2 \mathbf{1} \right) \rho \quad (1.130)$$

so that  $\text{Tr } \mathbf{Q} = 0$ . See (2.46).

$$4\pi\mathbf{a}(\mathbf{r}, t) = \frac{1}{c} \frac{\int(\mathrm{d}\mathbf{r}') \mathbf{j}(\mathbf{r}', t - r/c)}{r} - \nabla \cdot \frac{1}{c} \frac{\int(\mathrm{d}\mathbf{r}') \mathbf{r}' \mathbf{j}(\mathbf{r}', t - r/c)}{r} + \dots \quad (1.134)$$

Two terms suffice to give the same degree of approximation as the first three terms in the scalar potential expansion. The integrals can be re-expressed in convenient form with the aid of the conservation equation,

$$\nabla \cdot \mathbf{j} + \frac{\partial}{\partial t} \rho = 0. \quad (1.135)$$

Multiplying (1.135) by  $\mathbf{r}$  and rearranging the terms, we obtain

$$\frac{\partial}{\partial t} \mathbf{r} \rho + \nabla \cdot (\mathbf{j} \mathbf{r}) = \mathbf{j}. \quad (1.136)$$

The process of volume integration, extended over the entire region occupied by the charge distribution, yields

$$\int(\mathrm{d}\mathbf{r}) \mathbf{j}(\mathbf{r}, t) = \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{d}(t). \quad (1.137)$$

Corresponding operations with  $\mathbf{r}$  replaced by the dyadic  $\mathbf{r}\mathbf{r}$  give, successively,

$$\frac{\partial}{\partial t} \mathbf{r}\mathbf{r} \rho + \nabla \cdot (\mathbf{j}\mathbf{r}\mathbf{r}) = \mathbf{r}\mathbf{j} + \mathbf{j}\mathbf{r}, \quad (1.138a)$$

$$\int(\mathrm{d}\mathbf{r}) (\mathbf{r}\mathbf{j}(\mathbf{r}, t) + \mathbf{j}(\mathbf{r}, t)\mathbf{r}) = \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{Q}(t). \quad (1.138b)$$

Now,

$$\mathbf{r}\mathbf{j} = \frac{\mathbf{r}\mathbf{j} + \mathbf{j}\mathbf{r}}{2} + \frac{\mathbf{r}\mathbf{j} - \mathbf{j}\mathbf{r}}{2} = \frac{\mathbf{r}\mathbf{j} + \mathbf{j}\mathbf{r}}{2} - \frac{1}{2} \mathbf{1} \times (\mathbf{r} \times \mathbf{j}), \quad (1.139)$$

and therefore

$$4\pi\mathbf{a}(\mathbf{r}, t) = \frac{1}{c} \frac{\partial}{\partial t} \frac{\mathbf{d}(t - r/c)}{r} + \nabla \times \frac{\mathbf{m}(t - r/c)}{r} - \frac{1}{2c} \frac{\partial}{\partial t} \nabla \cdot \frac{\mathbf{Q}(t - r/c)}{r}, \quad (1.140)$$

where

$$\mathbf{m}(t) = \frac{1}{2c} \int(\mathrm{d}\mathbf{r}) \mathbf{r} \times \mathbf{j}(\mathbf{r}, t) \quad (1.141)$$

is the magnetic dipole moment of the system.

For a neutral molecule,  $q = 0$ , and the dominant term in the scalar potential expansion (1.132) is that associated with the electric dipole moment. The quadrupole moment contribution is smaller by a factor of the same magnitude as the larger of the two ratios  $a/\lambda$ ,  $a/r$ , and will be discarded. The electric dipole moment term predominates in the vector potential expansion (1.140) save for static or quasistatic phenomena when the magnetic dipole moment effect may assume importance. The quadrupole moment term will also be discarded here. Thus, under the conditions contemplated, the potentials can be expressed in terms of two vectors, the electric and magnetic Hertz vectors,

$$\mathbf{\Pi}_e(\mathbf{r}, t) = \frac{1}{4\pi r} \mathbf{d}(t - r/c), \quad (1.142a)$$

$$\mathbf{\Pi}_m(\mathbf{r}, t) = \frac{1}{4\pi r} \mathbf{m}(t - r/c), \quad (1.142b)$$

by

$$\phi(\mathbf{r}, t) = -\nabla \cdot \mathbf{\Pi}_e(\mathbf{r}, t), \quad (1.143a)$$

$$\mathbf{a}(\mathbf{r}, t) = \frac{1}{c} \frac{\partial}{\partial t} \mathbf{\Pi}_e(\mathbf{r}, t) + \nabla \times \mathbf{\Pi}_m(\mathbf{r}, t). \quad (1.143b)$$

The consistency of the approximations for the vector and scalar potentials is verified on noting that these expressions satisfy the Lorentz condition (this statement also applies to the discarded quadrupole moment terms). The electric and magnetic field intensities are given by

$$\mathbf{e} = \nabla \nabla \cdot \mathbf{\Pi}_e - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{\Pi}_e - \frac{1}{c} \frac{\partial}{\partial t} \nabla \times \mathbf{\Pi}_m, \quad (1.144a)$$

$$\begin{aligned} \mathbf{b} &= \nabla \times (\nabla \times \mathbf{\Pi}_m) + \frac{1}{c} \frac{\partial}{\partial t} \nabla \times \mathbf{\Pi}_e \\ &= \nabla \nabla \cdot \mathbf{\Pi}_m - \nabla^2 \mathbf{\Pi}_m + \frac{1}{c} \frac{\partial}{\partial t} \nabla \times \mathbf{\Pi}_e. \end{aligned} \quad (1.144b)$$

The Hertz vectors can be considered as the retarded solutions of the differential equations, because  $-\nabla^2 1/r = 4\pi\delta(\mathbf{r})$ ,

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{\Pi}_e(\mathbf{r}, t) = -\mathbf{d}(t)\delta(\mathbf{r}) \equiv -\mathbf{d}(\mathbf{r}, t), \quad (1.145a)$$

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{\Pi}_m(\mathbf{r}, t) = -\mathbf{m}(t)\delta(\mathbf{r}) \equiv -\mathbf{m}(\mathbf{r}, t). \quad (1.145b)$$

The fields associated with the Hertz vectors can be regarded as produced by point distributions of charge and current. Since

$$\begin{aligned} \nabla \times \mathbf{b} - \frac{1}{c} \frac{\partial}{\partial t} \mathbf{e} &= -\frac{1}{c} \frac{\partial}{\partial t} \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{\Pi}_e - \nabla \times \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{\Pi}_m \\ &= \frac{1}{c} \frac{\partial}{\partial t} \mathbf{p}(\mathbf{r}, t) + \nabla \times \mathbf{m}(\mathbf{r}, t) \end{aligned} \quad (1.146)$$

and

$$\nabla \cdot \mathbf{e} = \nabla \cdot \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{\Pi}_e = -\nabla \cdot \mathbf{d}(\mathbf{r}, t), \quad (1.147)$$

the required distributions are

$$\rho_{\text{eff}} = -\nabla \cdot \mathbf{d}(\mathbf{r}, t), \quad (1.148a)$$

$$\mathbf{j}_{\text{eff}} = \frac{\partial}{\partial t} \mathbf{p}(\mathbf{r}, t) + c \nabla \times \mathbf{m}(\mathbf{r}, t). \quad (1.148b)$$

Notice that the dipole moments of this effective distribution are those of the original molecule,

$$\begin{aligned} \int (\mathrm{d}\mathbf{r}) \mathbf{r} \rho_{\mathrm{eff}} &= - \int (\mathrm{d}\mathbf{r}) \mathbf{r} \nabla \cdot \mathbf{d}(\mathbf{r}, t) = \int (\mathrm{d}\mathbf{r}) \mathbf{d}(\mathbf{r}, t) \\ &= \mathbf{d}(t), \end{aligned} \quad (1.149\mathrm{a})$$

$$\begin{aligned} \frac{1}{2c} \int (\mathrm{d}\mathbf{r}) \mathbf{r} \times \mathbf{j}_{\mathrm{eff}} &= \frac{1}{2} \int (\mathrm{d}\mathbf{r}) \mathbf{r} \times [\nabla \times \mathbf{m}(\mathbf{r}, t)] = \int (\mathrm{d}\mathbf{r}) \mathbf{m}(\mathbf{r}, t) \\ &= \mathbf{m}(t). \end{aligned} \quad (1.149\mathrm{b})$$

Use has been made of the two identities

$$\mathbf{r} \nabla \cdot \mathbf{A} = \nabla \cdot (\mathbf{A}\mathbf{r}) - \mathbf{A}, \quad (1.150\mathrm{a})$$

$$\mathbf{r} \times (\nabla \times \mathbf{A}) = \nabla (\mathbf{A} \cdot \mathbf{r}) - \nabla \cdot (\mathbf{r}\mathbf{A}) + 2\mathbf{A}, \quad (1.150\mathrm{b})$$

and of the fact that

$$\int (\mathrm{d}\mathbf{r}) \mathbf{r} \delta(\mathbf{r}) = 0. \quad (1.151)$$

Therefore, the actual charge-current distribution in the molecule can be replaced by the effective point distribution without altering the values of the moments, or of the field at a sufficient distance from the molecule ( $r \gg a$ ,  $\lambda \gg a$ ).

Although the fields deduced from the effective distribution do not agree with the actual fields in the neighborhood of the molecule, nevertheless certain average properties of the fields are correctly represented. We shall show that the field intensities averaged over the volume contained in a sphere that includes the molecule, but is small in comparison with all wavelengths, is given correctly by the fields calculated from the effective distribution. It will follow, a fortiori, that the same property is maintained for any larger region of integration. In the immediate vicinity of the molecule, the potentials can be calculated, to a first approximation, by ignoring the finite propagation velocity of light,

$$\phi(\mathbf{r}, t) = \int (\mathrm{d}\mathbf{r}') \frac{\rho(\mathbf{r}', t)}{4\pi|\mathbf{r} - \mathbf{r}'|}, \quad (1.152\mathrm{a})$$

$$\mathbf{a}(\mathbf{r}, t) = \frac{1}{c} \int (\mathrm{d}\mathbf{r}') \frac{\mathbf{j}(\mathbf{r}', t)}{4\pi|\mathbf{r} - \mathbf{r}'|}. \quad (1.152\mathrm{b})$$

The fields are, correspondingly,

$$\mathbf{e}(\mathbf{r}, t) = -\nabla \int (\mathrm{d}\mathbf{r}') \frac{\rho(\mathbf{r}', t)}{4\pi|\mathbf{r} - \mathbf{r}'|} = \int (\mathrm{d}\mathbf{r}') \rho(\mathbf{r}', t) \nabla' \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}, \quad (1.153\mathrm{a})$$

$$\mathbf{b}(\mathbf{r}, t) = \nabla \times \frac{1}{c} \int (\mathrm{d}\mathbf{r}') \frac{\mathbf{j}(\mathbf{r}', t)}{4\pi|\mathbf{r} - \mathbf{r}'|} = \frac{1}{c} \int (\mathrm{d}\mathbf{r}') \mathbf{j}(\mathbf{r}', t) \times \nabla' \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}. \quad (1.153\mathrm{b})$$

The vector potential contribution to the electric field has been discarded in comparison with the electrostatic field. These fields are to be integrated over the volume  $V_S$  of a sphere which includes the entire molecule. The center of the sphere bears no necessary relation to the molecule. The integration for both fields requires an evaluation of

$$\int_{V_S} \frac{(\mathbf{dr})}{4\pi|\mathbf{r}-\mathbf{r}'|}, \quad (1.154)$$

extended over the sphere. On remarking that

$$\nabla'^2 \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = -\delta(\mathbf{r}-\mathbf{r}') \quad (1.155)$$

as the static limit of the differential equation satisfied by  $G(\mathbf{r})$ , (1.107), it is observed that

$$\nabla'^2 \int_{V_S} \frac{(\mathbf{dr})}{4\pi|\mathbf{r}-\mathbf{r}'|} = -1, \quad (1.156)$$

provided that the point  $\mathbf{r}'$  is within the sphere, as required by the assumption that the sphere encompasses the entire molecule. If the origin of coordinates is temporarily moved to the center of the sphere, it may be inferred that

$$\int_{V_S} \frac{(\mathbf{dr})}{4\pi|\mathbf{r}-\mathbf{r}'|} = -\frac{r'^2}{6} + \psi(\mathbf{r}'), \quad (1.157)$$

where  $\psi(\mathbf{r}')$  is a solution of Laplace's equation,

$$\nabla'^2 \psi(\mathbf{r}') = 0, \quad (1.158)$$

which, by symmetry, can depend only on the distance to the center of the sphere. Such a function must be a constant, for on integration of Laplace's equation over a sphere of radius  $r$ , and employing the divergence theorem, one obtains

$$4\pi r^2 \frac{d}{dr} \psi(r) = 0, \quad (1.159)$$

which establishes the constancy of  $\psi(\mathbf{r})$  within the sphere. Therefore,

$$\nabla' \int_{V_S} \frac{(\mathbf{dr})}{4\pi|\mathbf{r}-\mathbf{r}'|} = -\frac{1}{3}\mathbf{r}', \quad (1.160)$$

which is independent of the radius of the sphere. It immediately follows from (1.153a) and (1.153b) that

$$\int_{V_S} (\mathbf{dr}) \mathbf{e}(\mathbf{r}, t) = -\frac{1}{3} \int (\mathbf{dr}') \mathbf{r}' \rho(\mathbf{r}', t), \quad (1.161a)$$

$$\int_{V_S} (\mathbf{dr}) \mathbf{b}(\mathbf{r}, t) = \frac{1}{3c} \int (\mathbf{dr}') \mathbf{r}' \times \mathbf{j}(\mathbf{r}', t), \quad (1.161b)$$

The vector  $\mathbf{r}'$  is referred to the center of the sphere as origin. To return to the origin of coordinates established at the center of the molecule,  $\mathbf{r}'$  must be replaced by  $\mathbf{r}' + \mathbf{R}$ , where  $\mathbf{R}$  is the position vector of the center of the molecule relative to the center of the sphere. Finally, then, using (1.137),

$$\int_{V_S} (d\mathbf{r}) \mathbf{e}(\mathbf{r}, t) = -\frac{1}{3} \mathbf{d}(t), \quad (1.162a)$$

$$\int_{V_S} (d\mathbf{r}) \mathbf{b}(\mathbf{r}, t) = \frac{2}{3} \mathbf{m}(t) + \frac{1}{3c} \mathbf{R} \times \dot{\mathbf{d}}(t), \quad (1.162b)$$

provided that the molecule is electrically neutral. The essential result of this calculation is that the volume integrals depend only upon the moments of the system, not upon the detailed charge-current distribution. Now the effective point distribution (1.148a), (1.148b) predicts the correct values of the moments, and must therefore lead to the same integrated field intensities.

The explicit calculation of the fields derived by (1.144a), (1.144b) from the electric Hertz vector (1.142a) gives

$$4\pi\mathbf{e} = \left( \frac{3\mathbf{r}\mathbf{r} \cdot \mathbf{d}}{r^5} - \frac{\mathbf{d}}{r^3} \right) + \frac{1}{c} \left( \frac{3\mathbf{r}\mathbf{r} \cdot \dot{\mathbf{d}}}{r^4} - \frac{\dot{\mathbf{d}}}{r^2} \right) + \frac{1}{c^2} \frac{\mathbf{r} \times (\mathbf{r} \times \ddot{\mathbf{d}})}{r^3}, \quad (1.163a)$$

$$4\pi\mathbf{b} = -\frac{1}{c} \frac{\mathbf{r} \times \dot{\mathbf{d}}}{r^3} - \frac{1}{c^2} \frac{\mathbf{r} \times \ddot{\mathbf{d}}}{r^2}. \quad (1.163b)$$

The electric dipole moment and its time derivative are to be evaluated at the retarded time  $t - r/c$ . The relative orders of magnitude of the three types of terms in the electric field are determined by the ratio  $r/\lambda$ . For  $r/\lambda \ll 1$  (but  $r/a \gg 1$ ), the electric field is essentially that of a static dipole. However, if  $r/\lambda \gg 1$ , the last term in both fields predominates, and

$$4\pi\mathbf{e} = \frac{1}{c^2} \frac{\mathbf{r} \times (\mathbf{r} \times \ddot{\mathbf{d}})}{r^3} = \frac{1}{c^2} \frac{\mathbf{n} \times (\mathbf{n} \times \ddot{\mathbf{d}})}{r}, \quad (1.164a)$$

$$4\pi\mathbf{b} = -\frac{1}{c^2} \frac{\mathbf{r} \times \ddot{\mathbf{d}}}{r^2} = -\frac{1}{c^2} \frac{\mathbf{n} \times \ddot{\mathbf{d}}}{r}, \quad (1.164b)$$

where

$$\mathbf{n} = \frac{\mathbf{r}}{r} \quad (1.165)$$

is a radial unit vector. Note that at these large distances, the electric and magnetic fields are transverse to the direction of observation and to each other, and equal in magnitude,

$$\mathbf{e} = \mathbf{b} \times \mathbf{n}, \quad \mathbf{b} = \mathbf{n} \times \mathbf{e}. \quad (1.166)$$

Therefore, the energy flux vector

$$\mathbf{S} = c \mathbf{e} \times \mathbf{b} = \mathbf{n} c e^2 = \frac{\mathbf{n}}{4\pi r^2} \frac{(\mathbf{n} \times \ddot{\mathbf{d}})^2}{4\pi c^3} \quad (1.167)$$

is directed radially outward from the molecule and the net amount of energy which leaves a sphere of radius  $r$  per unit time is

$$P = \oint dS \mathbf{n} \cdot \mathbf{S} = \int_0^\pi \frac{2\pi r^2 \sin \theta d\theta}{4\pi r^2} \frac{(\ddot{\mathbf{d}})^2}{4\pi c^3} \sin^2 \theta = \frac{2}{3c^3} \frac{1}{4\pi} (\ddot{\mathbf{d}})^2. \quad (1.168)$$

This expression is independent of the radius of the sphere, except insofar as the radius  $r$  determines the time of emission of the field under observation, and represents the rate at which the molecule loses energy by radiation. In the particular situation of a dipole moment that oscillates harmonically with a single frequency,

$$\mathbf{d}(t) = \mathbf{d}_0 \cos \omega t, \quad (1.169)$$

the rate of emission of energy is

$$P = \frac{2}{3c^3} \frac{\omega^4}{4\pi} (\mathbf{d}_0)^2 \cos^2 \omega(t - r/c), \quad (1.170)$$

which fluctuates about the average value

$$\bar{P} = \frac{\omega^4}{3c^3} \frac{1}{4\pi} (\mathbf{d}_0)^2. \quad (1.171)$$

The fields generated by a magnetic dipole moment can be obtained from the electric dipole fields by the substitutions<sup>5</sup>

$$\mathbf{d} \rightarrow \mathbf{m}, \quad \mathbf{e} \rightarrow \mathbf{b}, \quad \mathbf{b} \rightarrow -\mathbf{e}, \quad (1.172)$$

that is

$$4\pi \mathbf{b} = \left( \frac{3\mathbf{r}\mathbf{r} \cdot \mathbf{m}}{r^5} - \frac{\mathbf{m}}{r^3} \right) + \frac{1}{c} \left( \frac{3\mathbf{r}\mathbf{r} \cdot \dot{\mathbf{m}}}{r^4} - \frac{\dot{\mathbf{m}}}{r^2} \right) + \frac{1}{c^2} \frac{\mathbf{r} \times (\mathbf{r} \times \ddot{\mathbf{m}})}{r^3}, \quad (1.173a)$$

$$4\pi \mathbf{e} = \frac{1}{c} \frac{\mathbf{r} \times \dot{\mathbf{m}}}{r^3} + \frac{1}{c^2} \frac{\mathbf{r} \times \ddot{\mathbf{m}}}{r^2}, \quad (1.173b)$$

and correspondingly, the rate of radiation is

$$P = \frac{2}{3c^3} \frac{1}{4\pi} (\ddot{\mathbf{m}})^2. \quad (1.174)$$

<sup>5</sup> See (1.144a) and (1.144b), and the fact that the Hertz vectors satisfy the wave equation away from the origin, (1.145a) and (1.145b). This symmetry is an example of electromagnetic duality, which is further explored in Problems 1.23, 1.25.

If oscillating electric and magnetic dipoles are present simultaneously, the total energy radiated per unit time is the sum of the individual radiation rates, for the outward energy flux is

$$\mathbf{n} \cdot \mathbf{S} = ce^2 = \frac{1}{4\pi r^2} \frac{1}{4\pi c^3} \left[ (\mathbf{n} \times \ddot{\mathbf{d}})^2 + (\mathbf{n} \times \ddot{\mathbf{m}})^2 + 2\mathbf{n} \cdot (\ddot{\mathbf{d}} \times \ddot{\mathbf{m}}) \right], \quad (1.175)$$

and the interference term disappears on integration over all directions of emission.

### 1.5.2 Work Done by Charges

It is instructive to calculate the rate of radiation by a system in a quite different manner, which involve evaluating the rate at which the charges in the molecule do work on the field and thus supply the energy which is dissipated in radiation. The precise statement of the consequence of energy conservation is obtained from (1.44a)

$$- \int_V (\mathbf{dr}) \mathbf{e} \cdot \mathbf{j} = \frac{d}{dt} E + P, \quad (1.176)$$

in which the integration is extended over a region  $V$  encompassing the molecule, and from (1.20a)

$$E = \int_V (\mathbf{dr}) \frac{e^2 + b^2}{2} \quad (1.177)$$

is the total electromagnetic energy associated with the molecule, while from (1.20b), the integral extended over the surface  $S$  bounding  $V$

$$P = \oint_S dS \mathbf{n} \cdot c \mathbf{e} \times \mathbf{b} \quad (1.178)$$

is the desired amount of energy leaving the system per unit time. This approach to the problem has the advantage of determining  $E$  and  $P$  simultaneously. In the evaluation of  $\int (\mathbf{dr}) \mathbf{e} \cdot \mathbf{j}$ , we are concerned only with the fields within the region occupied by charge. The effect of retardation, or the finite speed of light, is slight and the difference between the charge density at the retarded time and at the local time can be expressed by a power series expansion, with  $1/c$  regarded as a small parameter,

$$\begin{aligned} \rho(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c) &= \rho(\mathbf{r}', t) - \frac{|\mathbf{r} - \mathbf{r}'|}{c} \frac{\partial}{\partial t} \rho(\mathbf{r}', t) \\ &+ \frac{|\mathbf{r} - \mathbf{r}'|^2}{2c^2} \frac{\partial^2}{\partial t^2} \rho(\mathbf{r}', t) - \frac{|\mathbf{r} - \mathbf{r}'|^3}{6c^3} \frac{\partial^3}{\partial t^3} \rho(\mathbf{r}', t) + \dots \end{aligned} \quad (1.179)$$

Hence, from (1.116),

$$\begin{aligned}
4\pi\phi(\mathbf{r}, t) &= \int (\mathrm{d}\mathbf{r}') \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{2c^2} \frac{\partial^2}{\partial t^2} \int (\mathrm{d}\mathbf{r}') |\mathbf{r} - \mathbf{r}'| \rho(\mathbf{r}', t) \\
&\quad - \frac{1}{6c^3} \frac{\partial^3}{\partial t^3} \int (\mathrm{d}\mathbf{r}') |\mathbf{r} - \mathbf{r}'|^2 \rho(\mathbf{r}', t) + \dots, \quad (1.180)
\end{aligned}$$

on employing charge conservation to discard the second term in the expansion. To the same order of approximation it is sufficient to write from (1.117)

$$\begin{aligned}
4\pi\mathbf{a}(\mathbf{r}, t) &= \frac{1}{c} \int (\mathrm{d}\mathbf{r}') \frac{\mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{c^2} \frac{\mathrm{d}}{\mathrm{d}t} \int (\mathrm{d}\mathbf{r}') \mathbf{j}(\mathbf{r}', t) \\
&= \frac{1}{c} \int (\mathrm{d}\mathbf{r}') \frac{\mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{c^2} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathbf{d}(t), \quad (1.181)
\end{aligned}$$

which uses (1.137), for we have consistently retained terms of the order  $1/c^3$  in the electric field intensity, from (1.48)

$$\begin{aligned}
4\pi\mathbf{e}(\mathbf{r}, t) &= -\nabla \int (\mathrm{d}\mathbf{r}') \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{2c^2} \frac{\partial^2}{\partial t^2} \int (\mathrm{d}\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \rho(\mathbf{r}', t) \\
&\quad + \frac{1}{3c^3} \frac{\partial^3}{\partial t^3} \int (\mathrm{d}\mathbf{r}') (\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}', t) - \frac{1}{c^2} \frac{\partial}{\partial t} \int (\mathrm{d}\mathbf{r}') \frac{\mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} \\
&\quad + \frac{1}{c^3} \frac{\mathrm{d}^3}{\mathrm{d}t^3} \mathbf{d}(t). \quad (1.182)
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{\partial}{\partial t} \int (\mathrm{d}\mathbf{r}') (\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}', t) &= \mathbf{r} \frac{\mathrm{d}}{\mathrm{d}t} \int (\mathrm{d}\mathbf{r}') \rho(\mathbf{r}', t) - \frac{\mathrm{d}}{\mathrm{d}t} \int (\mathrm{d}\mathbf{r}') \mathbf{r}' \rho(\mathbf{r}', t) \\
&= -\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{d}(t) \quad (1.183)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t} \int (\mathrm{d}\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \rho(\mathbf{r}', t) &= - \int (\mathrm{d}\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \nabla' \cdot \mathbf{j}(\mathbf{r}', t) = - \int (\mathrm{d}\mathbf{r}') \frac{\mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} \\
&\quad + \int (\mathrm{d}\mathbf{r}') \frac{(\mathbf{r} - \mathbf{r}')(\mathbf{r} - \mathbf{r}') \cdot \mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|^3}, \quad (1.184)
\end{aligned}$$

whence

$$\begin{aligned}
4\pi\mathbf{e}(\mathbf{r}, t) &= -\nabla \int (\mathrm{d}\mathbf{r}') \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{2c^2} \frac{\partial}{\partial t} \int (\mathrm{d}\mathbf{r}') \left[ \frac{\mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} \right. \\
&\quad \left. + \frac{(\mathbf{r} - \mathbf{r}')(\mathbf{r} - \mathbf{r}') \cdot \mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|^3} \right] + \frac{2}{3c^3} \frac{\mathrm{d}^3}{\mathrm{d}t^3} \mathbf{d}(t). \quad (1.185)
\end{aligned}$$

Therefore, in (1.176) we encounter

$$\begin{aligned}
-4\pi \int (\mathrm{d}\mathbf{r}) \mathbf{j} \cdot \mathbf{e} &= \frac{\mathrm{d}}{\mathrm{d}t} \left[ \frac{1}{2} \int (\mathrm{d}\mathbf{r}) (\mathrm{d}\mathbf{r}') \frac{\rho(\mathbf{r}, t) \rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} \right. \\
&\quad + \frac{1}{4c^2} \int (\mathrm{d}\mathbf{r}) (\mathrm{d}\mathbf{r}') \left\{ \frac{\mathbf{j}(\mathbf{r}, t) \cdot \mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} \right. \\
&\quad \left. \left. + \frac{(\mathbf{r} - \mathbf{r}') \cdot \mathbf{j}(\mathbf{r}, t) (\mathbf{r} - \mathbf{r}') \cdot \mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|^3} \right\} \right] - \frac{2}{3c^3} \dot{\mathbf{d}}(t) \cdot \dot{\mathbf{d}}(t) ,
\end{aligned} \tag{1.186}$$

for

$$\begin{aligned}
\int (\mathrm{d}\mathbf{r}) \mathbf{j}(\mathbf{r}, t) \cdot \nabla \int (\mathrm{d}\mathbf{r}') \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} &= - \int (\mathrm{d}\mathbf{r}) (\mathrm{d}\mathbf{r}') \nabla \cdot \mathbf{j}(\mathbf{r}, t) \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int (\mathrm{d}\mathbf{r}) (\mathrm{d}\mathbf{r}') \frac{\rho(\mathbf{r}, t) \rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} ,
\end{aligned} \tag{1.187}$$

and we have used (1.137) again. As a last rearrangement,

$$-\frac{2}{3c^3} \dot{\mathbf{d}} \cdot \dot{\mathbf{d}} = -\frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{2}{3c^3} \dot{\mathbf{d}} \cdot \dot{\mathbf{d}} \right) + \frac{2}{3c^3} (\ddot{\mathbf{d}})^2 . \tag{1.188}$$

The integral  $-\int (\mathrm{d}\mathbf{r}) \mathbf{j} \cdot \mathbf{e}$  has thus been expressed in the desired form (1.176) as the time derivative of a quantity plus a positive definite expression which is to be identified with the rate of radiation,

$$P = \frac{2}{3c^3} \frac{1}{4\pi} (\ddot{\mathbf{d}})^2 . \tag{1.189}$$

Correct to terms of order  $1/c^2$  the electromagnetic energy of the molecule is<sup>6</sup>

$$\begin{aligned}
E &= \frac{1}{2} \int (\mathrm{d}\mathbf{r}) (\mathrm{d}\mathbf{r}') \frac{\rho(\mathbf{r}, t) \rho(\mathbf{r}', t)}{4\pi |\mathbf{r} - \mathbf{r}'|} \\
&\quad + \frac{1}{4c^2} \int (\mathrm{d}\mathbf{r}) (\mathrm{d}\mathbf{r}') \left( \frac{\mathbf{j}(\mathbf{r}, t) \cdot \mathbf{j}(\mathbf{r}', t)}{4\pi |\mathbf{r} - \mathbf{r}'|} + \frac{(\mathbf{r} - \mathbf{r}') \cdot \mathbf{j}(\mathbf{r}, t) (\mathbf{r} - \mathbf{r}') \cdot \mathbf{j}(\mathbf{r}', t)}{4\pi |\mathbf{r} - \mathbf{r}'|^3} \right) .
\end{aligned} \tag{1.191}$$

Magnetic dipole radiation first appears in that approximation which retains terms of the order  $1/c^5$ , with the expected result (1.174).

<sup>6</sup> It is the presence of the third term in (1.191) that results in the attraction between like currents, described by the magnetostatic energy

$$E = -\frac{1}{2c^2} \int (\mathrm{d}\mathbf{r}) (\mathrm{d}\mathbf{r}') \frac{\mathbf{J}(\mathbf{r}) \cdot \mathbf{J}(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|} . \tag{1.190}$$

See [9], Chap. 33.

## 1.6 Macroscopic Fields

The electromagnetic fields within material bodies, composed of enormous numbers of individual particles, are extremely complicated functions of position and time, on an atomic scale. Even if the formidable task of constructing the fields in a given situation could be performed, the description thus obtained would be unnecessarily elaborate, for it would contain information that could not be verified by our gross, macroscopic measuring instruments which respond only to the effects of many elementary particles. A macroscopic measurement of the instantaneous value of the field at a point is, in reality, a measurement of an average field within a region containing many atoms and extending over an interval of time large in comparison with atomic periods. It is natural, then, to seek an approximate form of the theory, so devised that the quantities which are the object of calculation are such averaged fields from which microscopic inhomogeneities have been removed, rather than the actual fields themselves. Such a program can be carried out if a length  $L$  and a time interval  $T$  exist which are small in comparison with distances and times in which macroscopic properties change appreciably, but large compared with atomic distances and times. These conditions are adequately satisfied under ordinary circumstances, failing only for matter of very low density or periodic fields of extremely short wavelength. Any quantity exhibiting enormous microscopic fluctuations, such as a field intensity, can be replaced by an microscopically smoothed quantity possessing only macroscopic variations by an averaging process conducted over a temporal interval  $T$  and a spatial region of linear extension  $L$ . Thus

$$\bar{f}(\mathbf{r}, t) = \frac{1}{V} \int_{|\mathbf{r}'| < L/2} (d\mathbf{r}') \frac{1}{T} \int_{-T/2}^{T/2} dt' F(\mathbf{r} + \mathbf{r}', t + t') \quad (1.192)$$

defines a space-time average of the function  $f(\mathbf{r}, t)$ , extended through the time interval from  $t - T/2$  to  $t + T/2$ , and over a spatial region of volume  $V$  which may be considered a sphere of diameter  $L$  drawn about the point  $\mathbf{r}$ . This averaging process has the important property expressed by

$$\overline{\nabla f(\mathbf{r}, t)} = \nabla \bar{f}(\mathbf{r}, t), \quad \overline{\frac{\partial}{\partial t} f(\mathbf{r}, t)} = \frac{\partial}{\partial t} \bar{f}(\mathbf{r}, t), \quad (1.193)$$

providing that the averaging domains are identical for all points of space and time. Hence, any linear differential equation connecting field variables can be replaced by formally identical equations for the averaged fields. Thus, in terms of the averaged field intensities,

$$\bar{\mathbf{e}}(\mathbf{r}, t) = \sqrt{\epsilon_0} \mathbf{E}(\mathbf{r}, t), \quad \bar{\mathbf{b}}(\mathbf{r}, t) = \frac{1}{\sqrt{\mu_0}} \mathbf{B}(\mathbf{r}, t), \quad (1.194)$$

the averaged Maxwell-Lorentz equations (1.19a), (1.19b) read

$$\nabla \times \frac{1}{\mu_0} \mathbf{B} = \frac{\partial}{\partial t} \epsilon_0 \mathbf{E} + \sqrt{\epsilon_0} \bar{\mathbf{j}}, \quad \nabla \cdot \epsilon_0 \mathbf{E} = \sqrt{\epsilon_0} \bar{\rho}, \quad (1.195a)$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0, \quad (1.195b)$$

in which we have introduced two new constants,  $\epsilon_0$  and  $\mu_0$ , related by

$$\epsilon_0 \mu_0 = \frac{1}{c^2}, \quad (1.196)$$

in order to facilitate the eventual adoption of a convenient system of units (SI) for macroscopic applications. Such averaged equations will be meaningful to the extent that they are independent of the precise size of the space-time averaging regions, within certain limits. This will be true if the sources of the macroscopic fields  $\mathbf{E}$  and  $\mathbf{B}$ , namely the averaged charge and current densities, can be expressed entirely in terms of the macroscopic field quantities and other large scale variables (temperature, density, etc.).

The actual charge distribution within a material medium arises not only from the charges within neutral atoms and molecules, which we shall call the bound charge, but also from relatively freely moving electrons (conduction electrons) and the charged atoms (ions) from which they have been removed.<sup>7</sup> The latter source of charge will be termed the free charge. We have already shown that the true bound charge-current distribution within a molecule can be replaced by an equivalent point distribution without affecting the values of integrated fields, or averaged fields, within a region large compared to the molecule. Hence, for the purpose of evaluating  $\bar{\rho}$  and  $\bar{\mathbf{j}}$ , the actual charge-current distribution can be written as the sum of a free charge distribution and the equivalent point distributions for the neutral molecules (and the ions, save for their net charge), given in (1.148a) and (1.148b),

$$\rho(\mathbf{r}, t) = \rho_f(\mathbf{r}, t) - \nabla \cdot \mathbf{d}(\mathbf{r}, t), \quad (1.197a)$$

$$\mathbf{j}(\mathbf{r}, t) = \mathbf{j}_f(\mathbf{r}, t) + \frac{\partial}{\partial t} \mathbf{d}(\mathbf{r}, t) + c \nabla \times \mathbf{m}(\mathbf{r}, t), \quad (1.197b)$$

where, summed over the molecules,

$$\mathbf{d}(\mathbf{r}, t) = \sum_a \mathbf{d}_a(t) \delta(\mathbf{r} - \mathbf{r}_a), \quad \mathbf{m}(\mathbf{r}, t) = \sum_a \mathbf{m}_a(t) \delta(\mathbf{r} - \mathbf{r}_a). \quad (1.198)$$

The averaged charge and current densities will be expressed, in the same form, in terms of averaged free charge and current densities and

$$\bar{\mathbf{d}}(\mathbf{r}, t) = n \bar{\mathbf{d}}, \quad \bar{\mathbf{m}}(\mathbf{r}, t) = n \bar{\mathbf{m}} \quad (1.199)$$

are the average dipole moments of a molecule within the smoothing region (in addition to the time average, a statistical average among the molecules is

<sup>7</sup> Holes in a semiconductor could also be contemplated.

implied), multiplied by the average density  $n$  of molecules at the macroscopic point in question. With the notation

$$\bar{\rho}_f(\mathbf{r}, t) = \frac{1}{\sqrt{\epsilon_0}} \rho(\mathbf{r}, t), \quad \bar{\mathbf{j}}_f(\mathbf{r}, t) = \frac{1}{\sqrt{\epsilon_0}} \mathbf{J}(\mathbf{r}, t), \quad (1.200a)$$

$$\bar{\mathbf{d}}(\mathbf{r}, t) = \frac{1}{\sqrt{\epsilon_0}} \mathbf{P}(\mathbf{r}, t), \quad \bar{\mathbf{m}}(\mathbf{r}, t) = \sqrt{\mu_0} \mathbf{M}(\mathbf{r}, t) \quad (1.200b)$$

for the macroscopic quantities measuring the free charge and current densities, and the electric and magnetic intensities of polarization (dipole moment per unit volume),<sup>8</sup> the averaged charge and current densities are

$$\sqrt{\epsilon_0} \bar{\rho} = \rho - \nabla \cdot \mathbf{P}, \quad (1.201a)$$

$$\sqrt{\epsilon_0} \bar{\mathbf{j}} = \mathbf{J} + \frac{\partial}{\partial t} \mathbf{P} + \nabla \times \mathbf{M}. \quad (1.201b)$$

Therefore, the first set of the averaged microscopic field equation (1.195a) – the Maxwell equations – read

$$\nabla \times \mathbf{H} = \frac{\partial}{\partial t} \mathbf{D} + \mathbf{J}, \quad \nabla \cdot \mathbf{D} = \rho, \quad (1.202a)$$

where

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}, \quad \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad (1.202b)$$

while the second set (1.195b) are unchanged. Thus the starting equations (1.1a), (1.1b) are recovered.

We also record the SI forms of the energy, energy flux vector, and momentum in vacuum ( $\mathbf{M} = \mathbf{P} = \mathbf{0}$ ):

$$U = \frac{1}{2} (\epsilon_0 E^2 + \mu_0 H^2), \quad (1.203a)$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \quad (1.203b)$$

$$\mathbf{G} = \mathbf{D} \times \mathbf{B}. \quad (1.203c)$$

The form of energy and momentum conservation in a medium is much more subtle, and will be treated subsequently.

## 1.7 Problems for Chap. 1

Note – In these problems, and in following chapters, we will use  $\mathbf{E}$ ,  $\mathbf{B}$ , and  $\mathbf{A}$  to denote the electric and magnetic fields, and the vector potential, both in macroscopic and microscopic situations, and we will use Heaviside-Lorentz

<sup>8</sup>  $\mathbf{P}$  and  $\mathbf{M}$  are also referred to as the electric polarization and the magnetization, respectively.

units for both, except in waveguide applications, which have a more engineering flavor. Again we remind the reader of the simple conversion factors necessary to pass between SI, Heaviside-Lorentz, and Gaussian units, described in the Appendix.

1. Verify the representations (1.7a) and (1.7b) for the three-dimensional delta function. Alternatively, derive them from the Fourier representations (1.94a) and (1.94b).
2. Establish the identities (1.22a) and (1.22b), and then prove the conservation statements (1.21) in empty space.
3. Prove the local statements of field energy and momentum non-conservation (1.44a) and (1.44b) from the inhomogeneous Maxwell equations (1.19a) and (1.19b).
4. Show that the energy is given by (1.75) by inserting (1.59) into (1.74); similarly, fill in the steps leading to (1.80) and (1.86).
5. Without reference to potentials, show that

$$-\square^2 \mathbf{E} = \left( -\nabla \rho - \frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{j} \right), \quad (1.204a)$$

$$-\square^2 \mathbf{B} = \frac{1}{c} \nabla \times \mathbf{j}. \quad (1.204b)$$

Here we have introduced the “d’Alembertian,” or wave operator,

$$\square^2 = -\frac{\partial^2}{c^2 \partial t^2} + \nabla^2. \quad (1.205)$$

Use the retarded solution of these equations to arrive at the asymptotic radiation fields of a bounded current distribution. (Don’t forget charge conservation,  $\frac{\partial}{\partial t} \rho + \nabla \cdot \mathbf{j} = 0$ .)

6. Starting from the Liénard-Wiechert potentials (1.125) and (1.126), work out  $\partial\tau/\partial t$  and  $\nabla\tau$  and so recognize that

$$\begin{Bmatrix} \phi \\ \mathbf{A} \end{Bmatrix}(\mathbf{r}, t) = \frac{q}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}(\tau)|} \begin{Bmatrix} \partial\tau/\partial t \\ \frac{1}{c} \partial\mathbf{r}(\tau)/\partial t \end{Bmatrix}. \quad (1.206)$$

Check that

$$(\nabla\tau)^2 - \left( \frac{1}{c} \frac{\partial}{\partial t} \tau \right)^2 = 0. \quad (1.207)$$

7. Work out the magnetic field of a moving point charge  $e$  by differentiating the  $\delta$ -function form for the potentials, (1.120a) and (1.120b). Get the radiation field part by considering only the derivative of the  $\delta$  function, and show that

$$\mathbf{B}(\mathbf{r}, t) \sim -\frac{e}{4\pi c^2} \frac{1}{|\mathbf{r} - \mathbf{r}(\tau)|} \mathbf{n} \times \frac{d^2 \mathbf{r}(\tau)}{dt^2}, \quad \text{where } \mathbf{n} = \frac{\mathbf{r} - \mathbf{r}(\tau)}{|\mathbf{r} - \mathbf{r}(\tau)|}. \quad (1.208)$$

Note carefully that  $d^2 \mathbf{r}(\tau)/dt^2$  is *not*  $d^2 \mathbf{r}(\tau)/d\tau^2$ .

8. What is the associated electric field as found by an analogous asymptotic computation from the potentials? Compare with (1.206).
9. Let  $\mathbf{E}_c$  be the instantaneous Coulomb field due to the charge density  $\rho$ . Demonstrate that (1.204a) can be presented as

$$-\square^2(\mathbf{E} - \mathbf{E}_c) = -\frac{1}{c^2} \frac{\partial}{\partial t} \left( \mathbf{j} + \frac{\partial}{\partial t} \mathbf{E}_c \right). \quad (1.209)$$

Begin with the retarded solution of this equation and derive the expressions for the electromagnetic energy and radiation power of a small current distribution.

10. Use the radiation fields derived above to compute the energy flux at large distances, per unit solid angle, for a point particle of charge  $e$  in terms of the acceleration and velocity of the particle at the retarded time – the emission time. Convert this energy per unit detection time into energy per unit emission time to get the power radiated into a given solid angle,

$$\frac{dP}{d\Omega} = \frac{e^2}{(4\pi)^2 c^3} \left[ \frac{\dot{\mathbf{v}}^2}{(1 - \mathbf{n} \cdot \mathbf{v}/c)^3} + 2 \frac{\mathbf{n} \cdot \dot{\mathbf{v}} \frac{1}{c} \mathbf{v} \cdot \dot{\mathbf{v}}}{(1 - \mathbf{n} \cdot \mathbf{v}/c)^4} - \left( 1 - \frac{v^2}{c^2} \right) \frac{(\mathbf{n} \cdot \dot{\mathbf{v}})^2}{(1 - \mathbf{n} \cdot \mathbf{v}/c)^5} \right]. \quad (1.210)$$

Show that this reduces to the formula for dipole radiation in the nonrelativistic limit,  $v/c \ll 1$ . For another expression of this result, see (3.111).

11. Integrate (1.210) over all directions to arrive at the expected result.
12. Use the result (1.210) to show, for the situation of linear acceleration, that is, when  $\dot{\mathbf{v}}$  is in the same direction as  $\mathbf{v}$ , which makes an angle  $\theta$  with respect to the direction of observation, that ( $\beta = v/c$ )

$$-\frac{d^2 E}{dt d\Omega} \Big|_{\text{rad}} = \frac{e^2}{(4\pi)^2 c^3} \left( \frac{d\mathbf{v}}{dt} \right)^2 \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}. \quad (1.211)$$

Integrate this over all solid angles to arrive at the energy loss rate for this circumstance.

13. Derive the dipole radiation formula for radiation emitted at a given frequency,

$$\frac{dE_{\text{rad}}}{d\omega} = \frac{2}{3\pi} \frac{1}{4\pi} \frac{1}{c^3} |\ddot{\mathbf{d}}(\omega)|^2. \quad (1.212)$$

Apply this formula to an instantaneous collision of two particles, one with mass  $m_1$  and charge  $e_1$ , the second with mass  $m_2$  and charge  $e_2$  in the center of mass frame (that is, the total momentum is zero). Let the angle of scattering of either particle be  $\theta$ . Ignoring radiation reaction, both particles have the same momentum magnitude  $p$  before and after the collision. What happens if  $e_1/m_1 = e_2/m_2$ ? From the photon viewpoint, how does the assumption that the kinetic energy of the particles is not changed restrict the radiation frequencies to which your result can be applied?

14. Consider a particle undergoing an instantaneous reversal in direction, changing from velocity  $\mathbf{v}$  to velocity  $-\mathbf{v}$  in negligible time. Derive the following formula for the number of photons with energy  $\hbar\omega$  emitted into a frequency interval  $d\omega$  and into an element of solid angle  $d\Omega$  making an angle  $\theta$  with respect to the direction specified by  $\mathbf{v}$ : ( $\alpha = e^2/4\pi\hbar c$ )

$$\frac{d^2N}{d\Omega d\omega} = \frac{\alpha}{4\pi^2} \frac{1}{\omega} \left\{ 2 \frac{1 + \beta^2}{1 - (\beta \cos \theta)^2} - (1 - \beta^2) \left[ \frac{1}{(1 - \beta \cos \theta)^2} + \frac{1}{(1 + \beta \cos \theta)^2} \right] \right\}. \quad (1.213)$$

15. What is the result of integrating (1.213) over all angles, for any  $\beta < 1$ . Does your photon spectrum agree with the known result for  $\beta \ll 1$ ? What does it become for  $\beta \approx 1$ ? Can you understand this result by looking at the approximate form derived from (1.213) for  $\beta \approx 1$ ,  $\theta \ll 1$ ,  $\pi - \theta \ll 1$ ?
16. Now suppose the charged particle stops on impact. Find the analog of the formula (1.213). Again, integrate it over all angles and look at the limits of  $\beta \ll 1$  and  $\beta \approx 1$ . Are the last two results what you would have expected? Explain.
17. Point charges  $e$  and  $-e$  are created at  $\mathbf{r} = \mathbf{0}$ ,  $t = 0$ , and then move with constant velocities  $\mathbf{v}$  and  $-\mathbf{v}$ , respectively. Derive the distribution in frequency and angle of the emitted radiation. Describe the angular distribution for  $v/c \approx 1$ . Repeat for one charge created at rest, the other with velocity  $\mathbf{v}$ .
18. Charge  $e$  is distributed uniformly over the surface of a sphere of radius  $a$ , which is rotating about an axis with constant angular velocity  $\omega$ . Compute the power radiated, either by applying a general method or by considering electric and magnetic dipole radiation.
19. A free electron at rest acted on by a light wave, and also the radiation reaction force, is described by

$$m\dot{\mathbf{v}} = e\text{Re} \mathbf{E} e^{-i\omega t} + \frac{1}{4\pi} \frac{2e^2}{3c^3} \ddot{\mathbf{v}}. \quad (1.214)$$

Solve this equation to get the total scattering cross section, defined as the ratio of the total power removed from the incident field,  $P_{\text{tot}}$ , to the incident flux,  $|\mathbf{S}|$ ,

$$\sigma_{\text{tot}} = \frac{P_{\text{tot}}}{|\mathbf{S}|}. \quad (1.215)$$

Express the cross section in terms of the so-called classical radius of the electron,  $r_0 = e^2/4\pi mc^2$  and the reduced wavelength  $\lambda = \lambda/(2\pi)$ . What is the limiting form for  $\lambda \ll r_0$ ?

20. Calculate the total cross section for the scattering of a plane wave by a dielectric sphere, assuming that the wavelength is large compared to the radius of the sphere.

21. Define a complex vector field by

$$\mathbf{F} = \mathbf{E} + i\mathbf{B}, \quad \mathbf{F}^* = \mathbf{E} - i\mathbf{B}. \quad (1.216)$$

Identify the scalar, vector, and dyadic, given by

$$\frac{1}{2}\mathbf{F}^* \cdot \mathbf{F}, \quad \frac{1}{2i}\mathbf{F}^* \times \mathbf{F}, \quad \text{and} \quad \frac{1}{2}(\mathbf{F}\mathbf{F}^* + \mathbf{F}^*\mathbf{F}), \quad (1.217)$$

respectively. What happens to these quantities if  $\mathbf{F}$  is replaced by  $e^{-i\phi}\mathbf{F}$ ,  $\phi$  being a constant?

22. What magnetic field is described, almost everywhere, by the vector potential

$$\mathbf{A}(\mathbf{r}) = \nabla \times \frac{g}{4\pi} \mathbf{n} \log(r - \mathbf{n} \cdot \mathbf{r}), \quad (1.218)$$

where  $g$  is a constant?

23. Consider Maxwell's equations with both electric (
- $\rho_e, \mathbf{j}_e$
- ) and magnetic (
- $\rho_m, \mathbf{j}_m$
- ) charges. Show that these equations retain their form under the electromagnetic rotation (duality transformation) under which electric (
- $\mathcal{E}$
- ) and magnetic (
- $\mathcal{M}$
- ) quantities are redefined according to

$$\mathcal{E} \rightarrow \mathcal{E} \cos \phi + \mathcal{M} \sin \phi, \quad \mathcal{M} \rightarrow \mathcal{M} \cos \phi - \mathcal{E} \sin \phi. \quad (1.219)$$

Check that the generalized Lorentz force

$$\mathbf{F} = e \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) + g \left( \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E} \right) \quad (1.220)$$

also retains its form under this rotation. Can you give a two-dimensional geometrical interpretation of the latter fact? A uni-directional electromagnetic pulse [recall the discussion after (1.33)] is characterized by the relations

$$E^2 - B^2 = 0, \quad \mathbf{E} \cdot \mathbf{B} = 0. \quad (1.221)$$

How do these properties respond to the electromagnetic rotation? For general electromagnetic fields, how do  $U$ ,  $\mathbf{G}$ , and  $\mathbf{T}$  respond to electromagnetic rotations?

24. From the Maxwell equations with both electric and magnetic charges considered in the previous problem, derive second order differential equations for
- $\mathbf{E}$
- and for
- $\mathbf{B}$
- . Show that

$$\mathbf{E} = -\nabla\phi_e - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}_e - \nabla \times \mathbf{A}_m, \quad (1.222a)$$

$$\mathbf{B} = -\nabla\phi_m - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}_m + \nabla \times \mathbf{A}_e. \quad (1.222b)$$

and exhibit the differential equations for these potentials in the Lorentz gauge, and in the radiation gauge.

25. Solve for the above potentials in some gauge, and find the asymptotic radiation field. Now what is the relationship between  $\mathbf{E}$  and  $\mathbf{B}$ ? Construct the spectral-angular distribution of the radiated power. How do it change under the duality transformation (1.219)?
26. A point magnetic charge  $g$  is at rest, at the origin. A point electric charge  $e$ , carried by a particle of mass  $m$  is in motion about the electric charge. What is the Newton-Lorentz equation of motion? By taking the moment of this equation, verify that the conserved angular momentum is

$$\mathbf{J} = \mathbf{r} \times m\mathbf{v} - \frac{eg}{4\pi c} \frac{\mathbf{r}}{r} . \quad (1.223)$$

What follows if quantum ideas about angular momentum are applied to the radial component of  $\mathbf{J}$ ?

27. Consider the relative motion of two particles with masses and electric and magnetic charges  $m_1, e_1, g_1$ , and  $m_2, e_2$ , and  $g_2$ , respectively. In deriving the equation of relative motion, which involves the reduced mass, remember that moving electric (magnetic) charges produce magnetic (electric) fields, but do not retain more than one factor of  $v_1/c$  or  $v_2/c$ . How do the combinations of  $e$ 's and  $g$ 's in this equation respond to the electromagnetic rotations of (1.219)? What is the conserved angular momentum?
28. A point magnetic charge  $g$  is located at the origin; a point electric charge  $e$  is located at the fixed point  $\mathbf{R}$ . What is the electromagnetic momentum density  $\mathbf{G}$  at an arbitrary position  $\mathbf{r}$ ? Write this vector as a curl. [This implies that  $\mathbf{G}$  is divergenceless; why is that?] Now construct the total electromagnetic angular momentum as the integrated moment of  $\mathbf{G}$ , simplified by partial integration. You will recognize the remaining integral as the electric field at  $\mathbf{R}$  produced by a charge density proportional to  $1/|\mathbf{r}|$ . Use spherical symmetry to solve the differential equation for the electric field (follow the known example of constant density). Compare your result with that of problem 26.
29. Consider Maxwell's equations in vacuum with both electric and magnetic charges and currents,  $\rho_e, \mathbf{j}_e, \rho_m$ , and  $\mathbf{j}_m$ . Write the similar Maxwell equations satisfied by

$$\mathbf{E}' = \mathbf{E} - \mathbf{E}_s , \quad \mathbf{B}' = \mathbf{B} - \mathbf{B}_s , \quad (1.224)$$

where  $\mathbf{E}_s$  and  $\mathbf{B}_s$  are the respective static fields at time  $t$  produced by the electric and magnetic charge densities at time  $t$ . That is,

$$\nabla \cdot \mathbf{B}_s(\mathbf{r}, t) = \rho_m(\mathbf{r}, t) , \quad \nabla \times \mathbf{B}_s(\mathbf{r}, t) = 0 , \quad (1.225)$$

and so on. What is  $\nabla \cdot \mathbf{E}'$ ,  $\nabla \cdot \mathbf{B}'$ ? Then what can you say about  $\mathbf{j}_e, \mathbf{j}_m$ , the currents that appear in the Maxwell equations obeyed by  $\mathbf{E}', \mathbf{B}'$ ? Use that property to redefine  $\mathbf{E}'$  so that you are left with the Maxwell equations without magnetic charge and current. Recognize that these fields can be constructed from a vector potential in the radiation gauge, and then exhibit  $\mathbf{E}$  and  $\mathbf{B}$ .

30. An electron moves at speed  $v \ll c$ , in a circular orbit of radius  $r$ , about an infinitely massive proton. Compute the rate of radiation – the rate of energy loss – first, in terms of  $v$  and  $r$ , and then in terms of the electron energy  $E$  (recall the virial theorem). Integrate the resulting differential equation for  $E$  to find the time it takes an electron, initially of energy  $E_0$ , to fall into the nucleus of this classical hydrogen atom. State the collapse time in seconds when the initial energy is that of the first Bohr orbit. (Here is one reason for inventing quantum mechanics.)
31. Demonstrate that

$$U_{\text{charges}} = \sum_a \delta(\mathbf{r} - \mathbf{r}_a(t)) E_a(t), \quad (1.226a)$$

$$\mathbf{S}_{\text{charges}} = \sum_a \delta(\mathbf{r} - \mathbf{r}_a(t)) E_a(t) \mathbf{v}_a(t), \quad (1.226b)$$

obey

$$\frac{\partial}{\partial t} U_{\text{ch}} + \nabla \cdot \mathbf{S}_{\text{ch}} = \mathbf{j} \cdot \mathbf{E}. \quad (1.227)$$

How does this lead to a direct proof of local total energy conservation? Proceed similarly with

$$\mathbf{G}_{\text{ch}} = \sum_a \delta(\mathbf{r} - \mathbf{r}_a) m_a \mathbf{v}_a, \quad (1.228a)$$

$$\mathbf{T}_{\text{ch}} = \sum_a \delta(\mathbf{r} - \mathbf{r}_a) m_a \mathbf{v}_a \mathbf{v}_a. \quad (1.228b)$$

32. Use the relativistic Lagrangian

$$L = -m_0 c^2 \sqrt{1 - \mathbf{v}^2/c^2} - e\phi + \frac{e}{c} \mathbf{v} \cdot \mathbf{A}, \quad \mathbf{v} = \frac{d\mathbf{r}}{dt}, \quad (1.229)$$

to deduce the Einstein-Lorentz equation of motion.

33. The inference of the fundamental field equations discloses that imparting a small velocity  $\delta\mathbf{v}$  to the system changes the fields by

$$\delta\mathbf{B} = \frac{\delta\mathbf{v}}{c} \times \mathbf{E}, \quad \delta\mathbf{E} = -\frac{\delta\mathbf{v}}{c} \times \mathbf{B}. \quad (1.230)$$

Show that Maxwell's equations, first without charge and current, retain their form if the meaning of the derivatives is also slightly altered:

$$\delta(\nabla) = -\frac{\delta\mathbf{v}}{c} \frac{1}{c} \frac{\partial}{\partial t}, \quad \delta\left(\frac{1}{c} \frac{\partial}{\partial t}\right) = -\frac{\delta\mathbf{v}}{c} \cdot \nabla. \quad (1.231)$$

Interpret this in terms of coordinate changes,  $\delta\mathbf{r}$ ,  $\delta t$ . [Hint:  $\nabla t = 0$ ,  $\partial\mathbf{r}/\partial t = 0$ .] Now show that all this remains true in the presence of charges, provided

$$\delta\mathbf{j} = \delta\mathbf{v}\rho, \quad \delta\rho = \frac{\delta\mathbf{v}}{c} \cdot \frac{1}{c} \mathbf{j}, \quad (1.232)$$

the first of which is expected. This is a first suggestion of the Lorentz transformations of Einstein relativity, which will be explored further in Chap. 3.

34. Consider the stress dyadic  $\mathbf{T}$  and the electromagnetic field of a unidirectional light pulse. Show that

$$\mathbf{T} \cdot \mathbf{E} = \mathbf{0}, \quad \mathbf{T} \cdot \mathbf{B} = \mathbf{0}, \quad \mathbf{T} \cdot \mathbf{E} \times \mathbf{B} = U \mathbf{E} \times \mathbf{B}. \quad (1.233)$$

Thus, in this situation,  $\mathbf{E} \times \mathbf{B}$  is an eigenvector of  $\mathbf{T}$  with the eigenvalue  $U$ ,  $\mathbf{E}$  and  $\mathbf{B}$  are eigenvectors with the eigenvalue zero. Are these properties consistent with  $\text{Tr } \mathbf{T} = U$ , (1.37)? What is the value of  $\det \mathbf{T}$  for the light pulse field?

35. Prove that the last result is unique to the light pulse by demonstrating, for an arbitrary field, that

$$\det \mathbf{T} = -U[U^2 - (c\mathbf{G})^2] \leq 0. \quad (1.234)$$

[Hint: find the eigenvalues of  $\mathbf{T}$ .] When does the equality sign hold? What is the value of  $\text{Tr } \mathbf{T}^2$ ?

36. Work out the three dimensional Coulomb Green's function in the vacuum,

$$G(\mathbf{r}) = \int \frac{(d\mathbf{k})}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2}, \quad (1.235)$$

by writing

$$\frac{1}{k^2} = \int_0^\infty d\lambda e^{-\lambda k^2}, \quad (1.236)$$

and then performing first the three integrations over the rectangular coordinates of  $\mathbf{k}$ . Repeat this calculation in four dimensions. Use your result to verify explicitly that

$$\int_{-\infty}^\infty dx_4 G(x_1, x_2, x_3, x_4) = G(\mathbf{r}). \quad (1.237)$$

Make this understandable by considering the four-dimensional differential equation that  $G(x_1, x_2, x_3, x_4)$  obeys.

37. Besides the advanced and retarded Green's functions considered in (1.106) et seq., another important Green's function is the casual or Feynman Green's function, defined by the 3 + 1 dimensional Fourier integral

$$G_+(\mathbf{r} - \mathbf{r}', t - t') = \int \frac{(d\mathbf{k})}{(2\pi)^3} \frac{d\omega}{2\pi} \frac{e^{i[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') - \omega(t-t')]} }{\mathbf{k}^2 - \omega^2/c^2 - i\epsilon}. \quad (1.238)$$

Evaluate this as

$$G_+ = \frac{ic}{4\pi^2} \frac{1}{(\mathbf{r} - \mathbf{r}')^2 - c^2(t - t')^2 + i\epsilon}. \quad (1.239)$$

Write the analogous definition and form of  $G_- = G_+^*$ . Check that

$$\frac{1}{2}(G_+ + G_-) = \frac{1}{2}(G_{\text{ret}} + G_{\text{adv}}), \quad (1.240)$$

where

$$G_{\text{ret,adv}} = \frac{\delta(t - t' \mp |\mathbf{r} - \mathbf{r}'|/c)}{4\pi|\mathbf{r} - \mathbf{r}'|}. \quad (1.241)$$

38. Solve the differential equation

$$(-\nabla^2 + \gamma^2)G(\mathbf{r}) = \delta(\mathbf{r}), \quad (1.242)$$

by Fourier transformation followed by a contour integration.