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Electromagnetic Radiation

Variational Methods, Waveguides, and Accelerators

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Electromagnetic Units

The question of electromagnetic units has been a vexing one for students of electromagnetic theory for generations, and is likely to remain so for the foreseeable future. It was thought by the reformers of the 1930s, Sommerfeld [29] and Stratton [30] in particular, that the rationalized system now encompassed in the standard Système International (SI) would supplant the older cgs systems, principally the Gaussian (G) and Heaviside-Lorentz (HL) systems. This has not occurred. This is largely because the latter are far more natural from a relativistic point of view; theoretical physicists, at least of the high-energy variety, use nearly exclusively rationalized or unrationalized cgs units. The advantages of the two mentioned cgs systems (there are other systems, which have completely fallen out of use) is that then electric and magnetic fields, **E**, **D**, **B**, **H** all have the same units, which is only natural since electric and magnetic fields transform into each other under Lorentz transformations. Electric permittivities and magnetic permeabilities correspondingly are dimensionless. The reason for the continued survival of two systems of cgs units lies in the question of "rationalization," that is, the presence or absence of 4πs in Maxwell's equations or in Coulomb's law. The rationalized Heaviside-Lorentz system is rather natural from a field theoretic point of view; but if one's interest is solely electromagnetism it is hard not to prefer Gaussian units.

In our previous book [9] we took a completely consistent approach of using Gaussian units throughout. However, such consistency is not present in any practitioner's work. Jackson's latest version of his classic text [10] changes horses midstream. Here we have adopted what may appear an even more schizophrenic approach: Where emphasis is on waveguide and transmission line descriptions, we use SI units, whereas more theoretical chapters are written in the HL system. This reflects the diverse audiences addressed by the materials upon which this book is based, engineers and physicists.

Thus we must live with disparate systems of electromagnetic units. The problem, however, is not so very complicated as it may first appear. Let us start by writing Maxwell's equations in an arbitrary system:

$$\nabla \cdot \mathbf{D} = k_1 \rho \, , \tag{1.1a}$$
$$\nabla \cdot \mathbf{B} = 0 \, , \tag{1.1b}$$
$$\nabla \times \mathbf{H} = k_2 \dot{\mathbf{D}} + k_1 k_2 \mathbf{J} \, , \tag{1.1c}$$
$$-\nabla \times \mathbf{E} = k_2 \dot{\mathbf{B}} \, , \tag{1.1d}$$

while the constitutive relations are

$$\mathbf{D} = k_3 \mathbf{E} + k_1 \mathbf{P} \, , \tag{1.2a}$$
$$\mathbf{H} = k_4 \mathbf{B} - k_1 \mathbf{M} \, . \tag{1.2b}$$

The Lorentz force law is

$$\mathbf{F} = e(\mathbf{E} + k_2 \mathbf{v} \times \mathbf{B}) \, . \tag{1.3}$$

The values of the four constants in the various systems of units are displayed in Table A.1. Here the constants appearing in the SI system have defined

Table A.1. Constants appearing in Maxwell's equations and the Lorentz force law in the different systems of units

constant	SI	HL	Gaussian
k_1	1	1	4π
k_2	1	$\frac{1}{c}$	$\frac{1}{c}$
k_3	ε_0	1	1
k_4	$\frac{1}{\mu_0}$	1	1

values:

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N\,A^{-2}} \, , \tag{1.4a}$$
$$\frac{1}{\sqrt{\epsilon_0 \mu_0}} = c \equiv 299\,792\,458\ \mathrm{m/s} \, , \tag{1.4b}$$

where the value of the speed of light is defined to be exactly the value given. (It is the presence of the arbitrary additional constant μ_0 which seems objectionable on theoretical grounds.)

Now we can ask how the various electromagnetic quantities are rescaled when we pass from one system of units to another. Suppose we take the SI system as the base. Then, in another system the fields and charges are given by

$$\mathbf{D} = \kappa_D \mathbf{D}^{\mathrm{SI}} \, , \quad \mathbf{E} = \kappa_E \mathbf{E}^{\mathrm{SI}} \, , \tag{1.5a}$$
$$\mathbf{H} = \kappa_H \mathbf{H}^{\mathrm{SI}} \, , \quad \mathbf{B} = \kappa_B \mathbf{B}^{\mathrm{SI}} \, , \tag{1.5b}$$
$$\mathbf{P} = \kappa_P \mathbf{P}^{\mathrm{SI}} \, , \quad \mathbf{M} = \kappa_M \mathbf{M}^{\mathrm{SI}} \, , \tag{1.5c}$$
$$\rho = \kappa_\rho \rho^{\mathrm{SI}} \, , \quad \mathbf{J} = \kappa_J \mathbf{J}^{\mathrm{SI}} \, , \tag{1.5d}$$

We insert these into the Maxwell equations, and determine the κs from the constants in Table A.1. For the Gaussian system, the results are

$$\kappa_D = \sqrt{\frac{\varepsilon_0}{4\pi}} \,, \tag{1.6a}$$

$$\kappa_E = \frac{1}{\sqrt{4\pi\varepsilon_0}} \,, \tag{1.6b}$$

$$\kappa_H = \frac{1}{\sqrt{4\pi\mu_0}} \,, \tag{1.6c}$$

$$\kappa_B = \sqrt{\frac{\mu_0}{4\pi}} \,, \tag{1.6d}$$

$$\kappa_P = \kappa_\rho = \kappa_J = \sqrt{4\pi\varepsilon_0} \,, \tag{1.6e}$$

$$\kappa_M = \sqrt{\frac{4\pi}{\mu_0}} \,. \tag{1.6f}$$

The conversion factors for HL units are the same except the various 4πs are omitted. By multiplying by these factors any SI equation can be converted to an equation in another system.

Here is a simple example of converting a formula. In SI, the skin depth of an imperfect conductor is given by (13.118),

$$\delta = \sqrt{\frac{2}{\mu\omega\sigma}} \,. \tag{1.7}$$

Converting into Gaussian units, the conductivity becomes

$$\sigma = \frac{J}{E} \to 4\pi\varepsilon\frac{J}{E} = 4\pi\sigma \,. \tag{1.8}$$

Therefore, the skin depth becomes

$$\delta \to \sqrt{\frac{2}{4\pi\varepsilon\mu\sigma\omega}} = \frac{c}{\sqrt{2\pi\sigma\omega}} \,, \tag{1.9}$$

which is the familiar Gaussian expression.

Let us illustrate how evaluation works in another simple example. The so-called classical radius of the electron is given in terms of the mass and charge on the electron, m and e, respectively,

$$r_0 = \left.\frac{e^2}{4\pi\varepsilon_0 mc^2}\right|_{\mathrm{SI}} = \left.\frac{e^2}{4\pi mc^2}\right|_{\mathrm{HL}} = \left.\frac{e^2}{mc^2}\right|_{\mathrm{G}} \,. \tag{1.10}$$

where the charges related by κ_ρ in (1.6e). Let us evaluate the formula in SI and G systems:

$$r_0 = \frac{(1.602 \times 10^{-19}\ \mathrm{C})^2 \times 10^{-7}\ \mathrm{N\,A^{-2}}}{9.109 \times 10^{-31}\ \mathrm{kg}} = 2.818 \times 10^{-15}\ \mathrm{m} \,, \tag{1.11a}$$

$$r_0 = \frac{4.803 \times 10^{-10}\ \mathrm{esu}}{9.109 \times 10^{-28}\ \mathrm{g} \times (2.998 \times 10^{10})^2\ \mathrm{cm/s}} = 2.818 \times 10^{-13}\ \mathrm{cm} \tag{1.11b}$$

It is even easier to evaluate this in terms of dimensionless quantities, such as the fine structure constant

$$\alpha = \left.\frac{e^2}{\hbar c}\right|_{\mathrm{G}} = \left.\frac{e^2}{4\pi\hbar c}\right|_{\mathrm{HL}} = \left.\frac{e^2}{4\pi\varepsilon_0\hbar c}\right|_{\mathrm{SI}} = \frac{1}{137.036} \,. \tag{1.12}$$

The classical radius of the electron is then proportional to the Compton wavelength of the electron,

$$\lambda_c = \frac{\hbar c}{mc^2} = 3.8616 \times 10^{-13}\ \mathrm{m}\,, \tag{1.13}$$

where a convenient conversion factor is $\hbar c = 1.97327 \times 10^{-5}$ eV cm. Thus

$$r_0 = \alpha\lambda_c = 2.818 \times 10^{-15}\ \mathrm{m}\,, \tag{1.14}$$

which incidentally shows that the "classical radius" gives an unphysically small measure of the "size" of an electron.

More discussion of electromagnetic units can be found in the Appendix of [9]. For a rather complete discussion see [31].