Appendix A

Units

This book has been written entirely in the Gaussian system of units, which is most convenient for theoretical purposes. However, for practical uses, and for engineering applications, it is essential to make contact with the SI (Système International) units, in which the meter, kilogram, second, and ampere are the fundamental units. Fortunately, it is very easy to transform equations written in the Gaussian system to the SI system. Here we will give that transformation, as well as explain the Gaussian units. Thus, evaluations may be performed in either system, or by using dimensionless quantities, such as the fine structure constant.

The macroscopic Maxwell equations, (4.60), in Gaussian units, are

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{D} &= 4\pi\rho, & \boldsymbol{\nabla}\cdot\mathbf{B} &= 0, \\ \boldsymbol{\nabla}\times\mathbf{H} &= \frac{1}{c}\frac{\partial}{\partial t}\mathbf{D} + \frac{4\pi}{c}\mathbf{J}, & -\boldsymbol{\nabla}\times\mathbf{E} &= \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}, \end{aligned} \tag{A.1}$$

where

$$\begin{aligned} \mathbf{D} &= \mathbf{E} + 4\pi\mathbf{P}, \\ \mathbf{H} &= \mathbf{B} - 4\pi\mathbf{M}. \end{aligned} \tag{A.2}$$

In addition, the Lorentz force law is

$$\mathbf{F} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right). \tag{A.3}$$

In contrast, the equations in the SI units are

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{D} &= \rho, & \boldsymbol{\nabla}\cdot\mathbf{B} &= 0, \\ \boldsymbol{\nabla}\times\mathbf{H} &= \frac{\partial}{\partial t}\mathbf{D} + \mathbf{J}, & -\boldsymbol{\nabla}\times\mathbf{E} &= \frac{\partial}{\partial t}\mathbf{B}, \end{aligned} \tag{A.4}$$

where

$$\begin{aligned} \mathbf{D} &= \epsilon_0\mathbf{E} + \mathbf{P}, \\ \mathbf{H} &= \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}. \end{aligned} \tag{A.5}$$

Here the constants ϵ_0 and μ_0 have defined values,

$$\begin{aligned}\epsilon_0\mu_0 &= c^{-2},\\ \mu_0 &= 4\pi \times 10^{-7},\end{aligned} \tag{A.6}$$

where the speed of light is now *defined* to be exactly

$$c = 299\,792\,458 \text{ m/s}. \tag{A.7}$$

In addition, the SI Lorentz force law is

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B}). \tag{A.8}$$

To convert from the equations in Gaussian form to those in SI form, we multiply each quantity in the former system by a suitable constant:

$$\begin{aligned}\mathbf{D} &\to k_D\mathbf{D}, \quad \mathbf{E} \to k_E\mathbf{E},\\ \mathbf{H} &\to k_H\mathbf{H}, \quad \mathbf{B} \to k_B\mathbf{B},\\ \mathbf{P} &\to k_P\mathbf{P}, \quad \mathbf{M} \to k_M\mathbf{M},\\ \rho &\to k_\rho\rho, \qquad \mathbf{J} \to k_J\mathbf{J}.\end{aligned} \tag{A.9}$$

Maxwell's equations, together with the relation between **D**, **E**, and **P**, and between **H**, **B**, and **M**, determine the ratios of all the k's; the Lorentz force law determines the overall scale. The results are

$$\begin{aligned}k_D &= \sqrt{\frac{4\pi}{\epsilon_0}}, \qquad k_E = \sqrt{4\pi\epsilon_0},\\ k_H &= \sqrt{4\pi\mu_0}, \quad k_B = \sqrt{\frac{4\pi}{\mu_0}},\\ k_\rho &= k_J = k_P = \frac{1}{\sqrt{4\pi\epsilon_0}},\\ k_M &= \sqrt{\frac{\mu_0}{4\pi}}.\end{aligned} \tag{A.10}$$

So this means that all that is necessary to convert a formula in Gaussian units into the corresponding formula in SI units is to make the scaling of the electromagnetic quantities by the factors given in (A.10).

To find the actual values of electromagnetic quantities in the two systems, such as the charge on the electron, it is further necessary to convert the length and mass units,

$$x \to \lambda_1 x, \quad m \to \lambda_2 m, \tag{A.11}$$

where here, of course, $\lambda_1 = 10^2$ and $\lambda_2 = 10^3$. Thus, for example, an energy scales by the factor $\lambda_1^2\lambda_2$, and hence the Coulomb potential of a electron at the origin is, in the two systems, related by

$$\frac{e_G^2}{r_G} = \lambda_1^2\lambda_2\frac{e_{\text{SI}}^2}{4\pi\epsilon_0 r_{\text{SI}}}, \tag{A.12}$$

Quantity	SI Unit	Corresponding number of Gaussian units
$\mathbf{E}$	V/m	$\frac{1}{3}\times 10^{-4}$
ϕ	V	$\frac{1}{300}$
e	C	3×10^{9}
ρ	C/m^3	3×10^{3}
I	A	3×10^{9}
$\mathbf{J}$	A/m^2	3×10^{5}
$\mathbf{P}$	C/m^2	3×10^{5}
$\mathbf{D}$	C/m^2	$12\pi\times 10^{5}$
σ	mho/m	9×10^{9} s^{-1}
C	F	9×10^{11} cm
$\mathbf{B}$	T	10^{4} gauss
$\mathbf{H}$	A-turn/m	$4\pi\times 10^{-3}$ oersted
$\mathbf{M}$	A-turn/m	10^{-3}
L	H	$\frac{1}{9}\times 10^{-11}$

Table A.1: Relation between SI electromagnetic units and Gaussian units. Here 3 is an approximation for 2.99792458.

where the subscripts now indicate the system of units. Thus

$$\begin{aligned} e_G &= e_{\mathrm{SI}}\frac{1}{\sqrt{4\pi\epsilon_0}}\sqrt{\frac{r_G}{r_{\mathrm{SI}}}}\sqrt{\lambda_1^2\lambda_2} \\ &= e_{\mathrm{SI}}\sqrt{10^{-7}}c\sqrt{10^2}\sqrt{10^7} \\ &= 1.602\times 10^{-19}\times 2.998\times 10^{9} = 4.803\times 10^{-10}\mathrm{esu}, \end{aligned} \tag{A.13}$$

where the name of the electrostatic unit is no longer of much significance. In this way we can easily work out the equivalence between SI units and their Gaussian counterparts, which are given in the Table.[1]

Finally, we turn to extraction of numbers from electromagnetic formulas, since this sometimes leads to confusion. We illustrate three different methods.

- The formula in Gaussian units may be evaluated directly.
- The formula may be transformed to SI units, using (A.10), and then evaluated using familiar SI quantities.
- The formula may be evaluated by eliminating electromagnetic quantities in favor of dimensionless numbers, and then the resulting formula is computed with the aid of dimensional analysis. This method, which may

[1] It might be noted that the Gaussian system is an amalgamation of the earlier cgs electrostatic and electromagnetic systems of units. Here esu units are used for electrical quantities and emu units are used for magnetic. Inductance sits on the fence: We use the esu unit, but some authors use the emu unit, which differs by a factor of $(3\times 10^{10})^{-2}$. A single factor of c gives the ratio of emu to esu units of charge. The other commonly used set of electrical units is the Heaviside-Lorentz system, which is "rationalized" like the SI system, the units differing, therefore, from the Gaussian units by various powers of 4π. See the reference at the end of this Appendix for more detail.

seem less direct, has the virtue that one need not keep track of factors of c, which is merely a conversion factor between time and space intervals, and is naturally set equal to unity in relativistic calculations.

We illustrate these three methods by evaluating the following formula, derived in Problem 32.1, for the mean lifetime of the orbital motion of an electron, having charge e and mass m, moving in a uniform magnetic field $\mathbf{B}$, whose motion decays due to the radiation emitted:

$$\tau = \frac{3m^3c^5}{4e^4B^2}. \tag{A.14}$$

Suppose $B = 10^4$ gauss. Then, in Gaussian units, where $m = 9.11 \times 10^{-28}$ g, $c = 3.00 \times 10^{10}$ cm/s, and $e = 4.80 \times 10^{-10}$ esu, this evaluates to $\tau = 2.58$ s. On the other hand, we may convert the formula first to SI:

$$\tau = \frac{3\pi\epsilon_0 m^3c^3}{e^4B^2}. \tag{A.15}$$

Now $m = 9.11 \times 10^{-31}$ kg, $c = 3.00 \times 10^8$ m/s, and $e = 1.60 \times 10^{-19}$ C, so for a magnetic field of 1T, we obtain, again, $\tau = 2.58$ s.

The dimensionless method consists, first, in replacing the electric charge by the fine structure constant,

$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}, \tag{A.16}$$

and the magnetic field is to be expressed in terms of its ratio to the characteristic (or critical) field strength,

$$B_c = \frac{m^2c^3}{e\hbar} = 4.41 \times 10^{13}\text{gauss}. \tag{A.17}$$

Thus our formula becomes

$$\tau = \frac{3}{4}\left(\frac{B_c}{B}\right)^2 \frac{\hbar}{\alpha m_e c^2}, \tag{A.18}$$

so, because $\hbar = 6.58 \times 10^{-22}$ MeV s, and $m_e c^2 = 0.511$ MeV, we get the same result, $\tau = 2.58$ s. Although this latter method may seem unnatural because it introduces quantum quantities into a classical calculation, it has the virtue of allowing the powers of $\hbar$ and c to be determined at the end of the calculation by elementary considerations of dimensional analysis. Here because mc^2 has dimensions of energy, exactly one factor of $\hbar$ must be present in the numerator to obtain a quantity having the dimensions of time.

This advantage is lost if one uses a purely classical method of eliminating e and B, for example, by eliminating the former in favor of the electron mass in terms of the classical electron radius, (45.6),

$$r_0 = \frac{e^2}{mc^2} = 2.818 \times 10^{-13}\text{cm}, \tag{A.19}$$

and expressing B in terms of the cyclotron frequency for a field B_0 of 1 gauss,

$$\omega_0 = \frac{eB_0}{mc} = 1.759 \times 10^7 \text{rad/s}, \tag{A.20}$$

so that our formula becomes

$$\tau = \frac{3}{4}\frac{c}{r_0\omega_0^2}\left(\frac{B_0}{B}\right)^2. \tag{A.21}$$

This, of course, gives the same answer for $B = 10^4$ gauss, namely 2.58 s, but now in terms of two dimensional quantities, r_0 and ω_0, instead of the one, m, in the atomic system. The latter is obviously preferable.

For a rather complete discussion of the vexing issue of electromagnetic units, the reader is referred to Francis B. Silsbee, *Systems of Electrical Units*, National Bureau of Standards Monograph 56 (U. S. Government Printing Office, Washington, 1962).