1. $$L = -m_0 c^2 \sqrt{1 - v^2/c^2} - \overbrace{\int (d\vec{r}) \left[\rho\varphi - \frac{1}{c}\vec{j}\cdot\vec{A} \right]}^{\sum e_a (\varphi(r_a) - \frac{\vec{v}_a}{c}\cdot\vec{A}(\vec{r}_a))}$$

$$+ \frac{1}{8\pi}\int (d\vec{r}) (E^2 - B^2)$$

a) $\delta\vec{r}$: $$\delta L = -m_0 c^2 \frac{1}{2}(1 - v^2/c^2)^{-1/2} 2\left(-\frac{\vec{v}}{c^2}\cdot\frac{d}{dt}\delta\vec{r}\right)$$

$$+ \frac{e\, d\delta\vec{r}}{c\, dt}\cdot\vec{A}(\vec{r}) - e\vec{\nabla}\left(\varphi - \frac{e}{c}\vec{v}\cdot\vec{A}\right)$$

$$= m_0 \vec{v}\gamma\cdot\frac{d\delta\vec{r}}{dt} + \frac{e}{c}\vec{A}\cdot\frac{d\delta\vec{r}}{dt}$$

$$e\vec{\nabla}\left[\varphi - \frac{e}{c}\vec{v}\cdot\vec{A}\right]$$

$$\delta W = \int_{t_0}^{t_1} dt\,\delta L = \vec{p}\cdot\delta\vec{r}\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} dt\,\delta\vec{r}\cdot\left[\frac{d}{dt}\vec{p} + e\vec{\nabla}\left(\varphi - \frac{1}{c}\vec{v}\cdot\vec{A}\right)\right]$$

$$\vec{p} = m_0\vec{v}\gamma + \frac{e}{c}\vec{A} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$\delta\vec{r}$ arb: $$\boxed{\frac{d\vec{p}}{dt} = -e\vec{\nabla}\left(\varphi - \frac{1}{c}\vec{v}\cdot\vec{A}\right)} \checkmark$$

b) or $$\frac{d}{dt}\left(m_0\vec{v}\gamma + \frac{e}{c}\vec{A}\right) = \frac{d}{dt} m_0\vec{v}\gamma + \frac{e}{c}\left(\frac{\partial}{\partial t}\vec{A} + \vec{v}\cdot\vec{\nabla}\vec{A}\right)$$

$$= -e\nabla\varphi + \frac{e}{c}\vec{\nabla}\vec{v}\cdot\vec{A}$$

or $$\boxed{\frac{d}{dt} m_0\vec{v}\gamma} = -e\nabla\varphi - \frac{e}{c}\frac{\partial}{\partial t}\vec{A} + \frac{e}{c}\left(\vec{v}\times(\vec{\nabla}\times\vec{A})\right)$$

$$\boxed{= e\vec{E} + e\frac{\vec{v}}{c}\times\vec{B}} \checkmark$$

c) $\delta\varphi$: $\delta L = \int (d\vec{r})\left[-\rho\,\delta\varphi + \frac{1}{4\pi}\vec{E}\cdot(-\vec{\nabla}\delta\varphi)\right]$

$$\underset{\text{int by part}}{=} \int (d\vec{r})\,\delta\varphi\left[-\rho + \frac{\vec{\nabla}\cdot\vec{E}}{4\pi}\right]$$

$\delta\varphi$ arb $\Rightarrow$ $\boxed{\vec{\nabla}\cdot\vec{E} = 4\pi\rho}$

d) $\delta\vec{A}$: $\delta L = \int (d\vec{r})\left[\frac{1}{c}\vec{j}\cdot\delta\vec{A} + \frac{\vec{E}}{4\pi}\cdot\left(-\frac{1}{c}\frac{\partial\,\delta\vec{A}}{\partial t}\right) - \frac{\vec{B}}{4\pi}\cdot\vec{\nabla}\times\delta\vec{A}\right]$

$$\delta W = \int_{t_0}^{t_1} dt\,\delta L = -\int (d\vec{r})\frac{\vec{E}}{4\pi c}\cdot\delta\vec{A}\Big|_{t_0}^{t_1}$$

$$+\int_{t_0}^{t_1} dt \int (d\vec{r})\,\delta\vec{A}\cdot\left[\frac{1}{c}\vec{j} + \frac{1}{4\pi c}\frac{\partial}{\partial t}\vec{E} - \frac{\vec{\nabla}\times\vec{B}}{4\pi}\right]$$

$\delta\vec{A}$ arb $\Rightarrow$ $\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \frac{4\pi}{c}\vec{j}$.

e) $t \to t + \delta t$, don't consider var. of fields:

$$\delta W = \int dt\,\frac{d\,\delta t}{dt}\left[L + \frac{\partial L}{\partial \vec{v}}\cdot(-\vec{v}) + \int \frac{\delta L}{\delta \vec{A}}\cdot\left(-\dot{\vec{A}}\right)(d\vec{r})\right]$$

(because $\vec{v} = \frac{d\vec{r}}{dt}$) $\left(\dot{\vec{A}} = \frac{\partial}{\partial t}\vec{A}\right)$

$$= -\delta t\, H \Big|_{t_0}^{t_1} + \int dt\, \delta t \frac{dH}{dt}$$

$$\Rightarrow \frac{dH}{dt} = 0 \,, \quad H = -L + \vec{v} \cdot \frac{\partial L}{\partial \vec{v}} + \int \dot{\vec{A}} \frac{\delta L}{\delta \dot{A}} (d\vec{r})$$

as stated

Explicitly, $\left(\frac{\partial L}{\partial \vec{v}} = \vec{p} = m_0 \gamma \vec{v} + \frac{e}{c} \vec{A} \right.$, as seen above$\left.\right)$

$$H = \vec{p} \cdot \vec{v} + \int (d\vec{r}) \Big[\frac{\vec{E}}{4\pi} \cdot \underbrace{\left(-\frac{1}{c} \dot{\vec{A}} \right)}_{\vec{E} + \nabla \varphi} \Big] - L$$

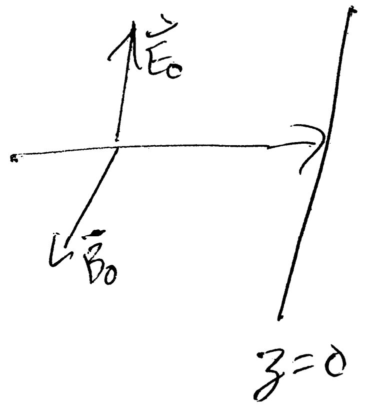
$$= m_0 \gamma \vec{v}^2 + m_0 c^2 \frac{1}{\gamma} + \int (d\vec{r}) \left(\frac{E^2}{8\pi} + \frac{B^2}{8\pi} \right)$$

$$\underbrace{- \int (d\vec{r}) \frac{\vec{\nabla} \cdot \vec{E}}{4\pi} \varphi + \int (d\vec{r}) \rho \varphi}_{0}$$

$$\underbrace{+ \frac{e}{c} \vec{A} \cdot \vec{v} - \frac{e}{c} \vec{A} \cdot \vec{v}}_{0}$$

$$\boxed{H = m_0 \gamma c^2 + \int d\vec{r} \left(\frac{E^2 + B^2}{8\pi} \right)}$$

2

a) 

We know $\vec{E} \times \vec{B} \propto \vec{k}$ ← direct. of propagation

$|\vec{E}| = |\vec{B}|$

So $|\vec{E}_0| = |\vec{B}_0|$, $\vec{E}_0 \perp \vec{B}_0$

& both are $\perp$ to z

Say $\vec{E}_0 = \hat{x} E_0$, $\vec{B}_0 = \hat{y} B_0$

Then Maxwell's eqns say

$$-\vec{\nabla} \times \vec{E} = \frac{1}{c}\frac{\partial}{\partial t}\vec{B}$$

or $-ikE_0 = \frac{1}{c}(-i\omega)B_0$, $E_0 = \frac{\omega}{ck} B_0$

& $\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial}{\partial t}\vec{E}$

or $-ikB_0 = -\frac{i\omega}{c}E_0$ $\quad B_0 = \frac{\omega}{ck} E_0$

$\therefore \omega^2 = c^2 k^2$, $\omega = \pm ck$, $B_0 = \pm E_0$

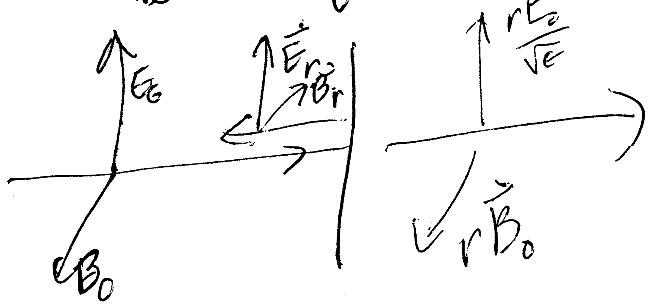
b)
Now at interface, there is reflection & transmission —   5

Reflected wave: $(z<0)$

$$\vec{B}(\vec{r},t) = r\,\vec{B}_0\, e^{-ikz-i\omega t}$$

$$\vec{E}(\vec{r},t) = -r\,\vec{E}_0\, e^{-ikz-i\omega t}$$

(because now $\vec{E}\times\vec{B} \propto -\hat{z}$, must reverse sense of either $\vec{E}$ or $\vec{B}$



Transmitted wave: $(z>0)$

$$\vec{B}(\vec{r},t) = t\,\vec{B}_0\, e^{ikz-i\omega t}$$

$$\vec{E}(\vec{r},t) = \frac{t\,\vec{E}_0}{\sqrt{\epsilon}}\, e^{ikz-i\omega t}$$

In dielectric medium, $|E| = \frac{|\vec{B}|}{\sqrt{\epsilon}}$

From ME: still have $E_0 = \frac{\omega}{ck} B_0$, but now $B_0 = \frac{\omega\epsilon E_0}{ck}$

so $\omega^2\epsilon = c^2k^2$, and $E_0 = \frac{1}{\sqrt{\epsilon}} B_0$

Continuity of $\vec{B}$ across interface:

$$\Rightarrow \quad 1+r=t$$

c) given above

d) $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$

just to left of interface $\vec{E} = \vec{E}_0(1-r)e^{-i\omega t}$

$$\vec{B} = \vec{B}_0(1+r)e^{-i\omega t}$$

$$\int dt\,\vec{S} = \frac{c}{4\pi}\int\frac{d\omega}{2\pi}E^*\times\vec{B}$$

$$= \frac{c}{4\pi}\hat{z}|B_0|^2(1-r^2)$$

or for given freq. $\frac{c}{4\pi}\hat{z}E_0^*B_0(1-r^2)$

$$[\ \mathrm{Re}\,\vec{E}_0e^{-i\omega t}\times\mathrm{Re}\,\vec{B}_0e^{-i\omega t}$$

$$= \frac{1}{4}(\vec{E}_0e^{-i\omega t}+\vec{E}_0^*e^{i\omega t})\times(\vec{B}_0e^{-i\omega t}+\vec{B}_0^*e^{+i\omega t})$$

$$= \frac{\hat{z}}{4}\left(E_0B_0e^{-2i\omega t}+E_0^*B_0^*e^{2i\omega t}+2\,\mathrm{Re}\,E_0^*B_0\right)$$

oscillate rapidly, average to zero

$$= \frac{\hat{z}}{2}|B_0|^2\ ]$$

To right of interface:

$$\frac{c}{4\pi}\hat{z}\,|B_0|^2\frac{t^2}{\sqrt{\epsilon}} = \frac{c}{4\pi}\hat{z}|B_0|^2\frac{(1+r)^2}{\sqrt{\epsilon}}$$

Note: this also follows from continuity of $E_\perp$: $1-r = \frac{t}{\sqrt{\epsilon}} = \frac{1+r}{\sqrt{\epsilon}}$

e) $\therefore$

$$(1-r)(1+r) = 1-r^2 = \frac{(1+r)^2}{\sqrt{\epsilon}} \Rightarrow (1-r)\sqrt{\epsilon} = 1+r$$

$$\text{or } \sqrt{\epsilon}-1 = r(\sqrt{\epsilon}+1)$$

$$r = \frac{\sqrt{\epsilon}-1}{\sqrt{\epsilon}+1}$$

$\epsilon \to 1 \quad r=0,\ t=1$

$\epsilon \to \infty \quad r=1,\ t=2$

But note there is no wave prop inside the medium, because the energy flux is proportional ~~to~~

$$\frac{t^2}{\sqrt{\epsilon}} \to \frac{2}{\sqrt{\epsilon}} \to 0.$$