Physics 5163
Statistical Mechanics
Homework 9
Due Friday, May 7, 2010 at 5pm

April 23, 2010

1. Quantum mechanically, derive the inverse transform that expresses the grand partition function, $X(\alpha, z)$, in terms of the structure function $\Omega(E, N)$. Then, supposing the system is composed of a large number n of intercommunicating but noninteracting subsystems, derive an asymptotic formula for the structure function using the saddle-point method. From this, derive the density operator for a single subsystem to be asymptotic to the grand canonical density operator,
$$\rho = \frac{e^{-\beta(H-\mu\mathcal{N})}}{X(\beta,\mu)},$$
where H is the Hamiltonian and $\mathcal{N}$ is the number operator.

2. As shown in class, for an ideal gas, the probability of finding N molecules in a system governed by the grand canonical distribution is given by the Poisson distribution:
$$P_N = \frac{1}{N!} e^{-\bar{N}} \bar{N}^N.$$
   (a) Show that
   $$\langle N \rangle = \bar{N}.$$
   (b) Show that the fluctuation in N for such a distribution is given by
   $$\sqrt{\langle (N-\bar{N})^2 \rangle} = \sqrt{\bar{N}}.$$

(c) Derive the same result by computing

$$\frac{1}{\beta^2}\frac{\partial^2}{\partial\mu^2}\ln X,$$

where X is the grand partition function for the ideal gas.

3. In this problem we will consider nonrelativistic fermions or bosons (for the latter, at temperatures above that for which Bose condensation occurs). Assume the spin degeneracy factor $g_s = 1$.

   (a) Show that the number of particles may be expressed in terms of the fugacity $\zeta = e^{\beta\mu}$ by

   $$N = \frac{V}{\lambda^3} g_{3/2}^{(\pm)}(\zeta), \quad \lambda = \frac{h}{(2\pi mkT)^{1/2}},$$

   where

   $$g_n^{(\pm)}(\zeta) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{dx\, x^{n-1}}{\zeta^{-1}e^x \pm 1}.$$

   (b) Derive a similar formula for the energy of the system, U.

   (c) Discuss the classical limit, $T \to \infty$, of the expression in parts a and b.

   (d) Prove the identity

   $$\frac{d}{d\zeta} g_n^{(\pm)}(\zeta) = \frac{1}{\zeta} g_{n-1}^{(\pm)}(\zeta)$$

   using integration by parts.

   (e) Using the fact that N is fixed, and the identity in part d, derive a formula for $d\zeta/dT$, the temperature derivative of the fugacity, in terms of $g_{3/2}^{(\pm)}(\zeta)$ and $g_{1/2}^{(\pm)}(\zeta)$.

   (f) Then obtain a formula for $c_v = \partial U/\partial T$ in terms of N, $g_{5/2}^{(\pm)}(\zeta)$, $g_{3/2}^{(\pm)}(\zeta)$, and $g_{1/2}^{(\pm)}(\zeta)$.

4. This problem concerns fluctuations in particle numbers.

(a) Show, generally, in a grand canonical ensemble, that

$$\langle (N - \overline{N})^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2 = \frac{1}{\beta^2} \frac{\partial^2}{\partial \mu^2} \ln X = \frac{1}{\beta} \left(\frac{\partial \overline{N}}{\partial \mu} \right)_{V,T},$$

where X is the grand partition function.

(b) From the thermodynamic relation

$$d\mu = -\frac{S}{N} dT + \frac{V}{N} dp$$

relate $\langle (N - \overline{N})^2 \rangle$ to the isothermal compressibility,

$$\kappa = -\frac{1}{V} \left(\frac{\partial V}{\partial p} \right)_{N,T}.$$

(c) Compute $\overline{N}$ for a photon gas (black-body radiation).

(d) What is $\langle (N - \overline{N})^2 \rangle$ for a photon gas? [Hint: What is p?]

(e) If n_i is the occupation number for the ith state of the system of bosons or fermions, show that in the grand canonical ensemble

$$\langle (n_i - \overline{n}_i)^2 \rangle = kT \frac{\partial}{\partial \mu} \langle n_i \rangle = \langle n_i \rangle (1 \pm \langle n_i \rangle).$$

(f) How is this result related to that given in part a? What about photons?

5. This problem sketches a crude model of a white dwarf star—a cold star supported by degenerate electron pressure.

(a) For a nonrelativistic fermi gas, compute the pressure p as a function of the number density, N/V, at zero temperature.

(b) For an ultrarelativistic fermi gas compute the pressure p as a function of N/V, at zero temperature.

(c) In the nonrelativistic regime, equate the force due to the pressure with that due to the gravitational potential on the surface:

$$p(R) 4\pi R^2 = \frac{GM^2}{R^2}.$$

From this relation determine the dependence of the radius R of the star upon its mass M. (Don't worry about numerical factors, but keep all physical constants.)

(d) In the ultrarelativistic limit, a unique value of M is determined by the above equation. This is the Chandrasekhar limiting mass (the upper limit to the mass of a white dwarf). What estimate do you obtain? Use the following rough values:

$$(\hbar c/G)^{1/2} \approx 10^{19}\text{GeV}/c^2, \quad m_p \approx 1\text{Gev}/c^2 \approx 2 \times 10^{-27}\text{kg},$$

and

$$M_\odot \approx 2 \times 10^{30}\text{kg}.$$

Problems in Pathria: 7.5, 7.6, 7.19, 8.4, 8.9.