Physics 5163. Homework 8
Due Wednesday, April 29, 2009

April 20, 2009

1. Consider a system composed of two gases, a and b. Prove that for two intercommunicating but noninteracting subsystems, the composition law for the structure function is

$$\frac{\Omega(E, N_a, N_b)}{N_a!\, N_b!} = \sum_{N_{a1}, N_{b1}} \int dE_1 \frac{\Omega_1(E_1, N_{a1}, N_{b1})}{N_{a1}!\, N_{b1}!} \times \frac{\Omega_2(E - E_1, N_a - N_{a1}, N_b - N_{b1})}{(N_a - N_{a1})!\, (N_b - N_{b1})!}.$$

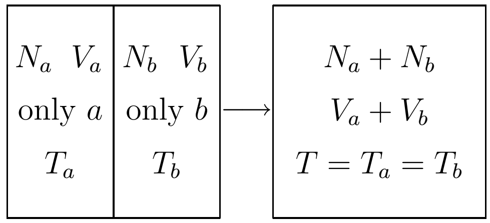
2. Show that for an arbitrary number of intercommunicating but non-interacting subsystems consisting of a single species of molecule, the generalized convolution law is

$$\frac{\Omega(E, N)}{N!} = \sum_{\{N_j\}} \delta_{N, \sum_j N_j} \int \delta(E - \sum_j E_j) \prod_j \frac{\Omega_j(E_j, N_j)}{N_j!} dE_j.$$

3. When there are two different species of particles, a and b, the entropy is given by

$$\frac{S}{k} = \ln \frac{\Omega(E, N_a, N_b)}{N_a!\, N_b!},$$

where N_i is the number of particles of the ith species. Consider the following situation pertaining to an ideal gas of such particles:



Initially, the particles are separated into distinct volumes V_a and V_b. Finally, they are mixed in a common volume $V = V_a + V_b$. The gases are in equilibrium before and after mixing, that is, $T_a = T_b = T$ and $N_a/V_a = N_b/V_b = (N_a + N_b)/(V_a + V_b)$. Calculate the change in entropy as a result of the mixing.

4. For a system composed of two weakly interacting parts which can exchange particle and energy with each other, in equilibrium not only are the temperatures of the two parts equal, but the chemical potentials as well. Prove this statement by requiring that the entropy S be an extremum with respect to the energy of one of the parts, U_1, as well as an extremum with respect to the particle number of one of the parts, N_1:
$$\frac{\partial S}{\partial U_1} = 0, \quad \frac{\partial S}{\partial N_1} = 0.$$

5. Now use the result of the previous problem to derive the "barometer" equation. That is, consider an ideal gas, for which the chemical potential is
$$\mu = kT \ln\left(\frac{N}{V}\frac{1}{(2m\pi kT)^{3/2}}\right).$$
Argue that the effect of gravity is to add a term mgz to μ; and thereby find the pressure p as a function of the altitude z.

6. Consider a system of n intercommunicating but noninteracting subsystems. If the whole system is governed by the microcanonical distribution, show that, if n is large, the structure function for one of the subsystems, which has N molecules and energy E, is asymptotic to
$$\frac{\Omega(E,N)}{N!} \sim e^{\beta E - \beta\mu N} X(\beta, \beta\mu) \frac{1}{2\pi\sqrt{\Delta}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right),$$

where

$$\Delta = \left[\left(\frac{\partial^2}{\partial \alpha^2} \ln X \right) \left(\frac{\partial^2}{\partial \nu^2} \ln X \right) - \left(\frac{\partial^2}{\partial \alpha \partial \nu} \ln X \right)^2 \right]_{\alpha=\beta, \nu=\beta\mu} .$$

Then show, up to terms of order $1/n$, that the single subsystem is governed by the grand canonical distribution,

$$\rho_{\mathrm{GC}}(N, x) = \frac{1}{N!} \frac{1}{X(\beta, \beta\mu)} e^{-\beta[H(x) - \mu N]},$$

where x represents a point in phase space.