Physics 5163
Statistical Mechanics
Second Examination

April 22, 2009

Instructions: Work all problems and show all your steps. If you have any questions, ask the instructor. If you get stuck on one part, assume an answer to continue on. Remember, this is a closed-book examination, so you may not consult books or notes. **Good Luck!**

1. In this problem we consider an atom with magnetic moment $\boldsymbol{\mu}$ in a magnetic field $\mathbf{B}$, for which system the Hamiltonian is

$$H = -\boldsymbol{\mu} \cdot \mathbf{B}.$$

Let $\mathbf{B}$ lie along the $\hat{\mathbf{z}}$ direction, $\mathbf{B} = B\hat{\mathbf{z}}$.

(a) *Classically*, calculate the partition function for this system. In this case, the angle θ between $\boldsymbol{\mu}$ and $\mathbf{B}$ is a continuous variable.

(b) From the partition function calculated in part a, compute the average magnetic moment along the $\hat{\mathbf{z}}$ direction,

$$\langle \mu_z \rangle = \langle \mu \cos\theta \rangle = \mu L(x), \quad x = \frac{\mu B}{kT},$$

that is, determine the Langevin function $L(x)$.

(c) Work out the behavior of $L(x)$ for $x \ll 1$ and for $x \gg 1$. Why is the latter result expected on physical grounds?

(d) *Quantum mechanically*, calculate the partition function for this system. In this case $\mu_z = g\mu_B m$, where $\mu_B = e\hbar/2mc$ is the Bohr magneton, g is the Landé g-factor (a given parameter), and m is a discrete variable which goes by integer steps from $-J$ to $+J$, where J is the total angular momentum quantum number of the atom.

(e) From the partition function calculated in part d, compute the average magnetic moment along the $\hat{\mathbf{z}}$ direction,

$$\langle \mu_z \rangle = \mu B_J(x), \quad x = \frac{\mu B}{kT}, \quad \mu = g\mu_B J,$$

where $B_J(x)$ is the so-called Brillouin function.

(f) Show that, as $J \to \infty$, the Brillouin function approaches the Langevin function,

$$B_J(x) \to L(x), \quad \text{as} \quad J \to \infty.$$

This is the correspondence-principle limit.

2. Consider a three-dimensional classical system consisting of a gas made up of N ultrarelativistic particles, for which the relation between the energy and the momentum is $E = pc$. Let the gas be contained within a volume V and ignore interactions.

(a) Calculate the partition function for this gas.

(b) Give an expression for the Helmholtz free energy F.

(c) Compute the internal energy U.

(d) Compute the entropy from

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}.$$

(e) Compute the pressure from

$$p = -\left(\frac{\partial F}{\partial V}\right)_{T,N},$$

and give the relation between p and the energy density U/V.

(f) Calculate the specific heat at constant volume, c_v, using the result of part c, and the specific heat at constant pressure, c_p, using the result of part e. What is the ratio

$$\gamma = \frac{c_p}{c_v}?$$

(g) Show, for an adiabatic process, that is, one for which S is constant, that $V^{\gamma-1}T$ is a constant.

(h) Finally, find the structure function $\Omega(E)$ from the partition function $Z(\alpha)$ by using the formula

$$\Omega(E) = \frac{1}{2\pi i}\int_C d\alpha\, e^{\alpha E} Z(\alpha).$$

What is the contour C? Use the Hankel representation of the Γ function,

$$\frac{1}{\Gamma(\beta)} = \frac{1}{2\pi i}\int_C dz\, e^z\, z^{-\beta}$$

to evaluate the integral.