Physics 5163
Statistical Mechanics
Final Examination
May 12, 2009

Instructions: Work all problems and show all your steps. If you have any questions, ask the proctor. If you get stuck on one part, assume an answer to continue on. Remember, this is a closed-book examination, so you may not consult books or notes. **Good Luck!**

1. Consider a one-dimensional harmonic oscillator, with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2,$$

   (a) *Classically*, what is the partition function $\chi(\beta)$? Give it as an explicit function of β and ω.

   (b) What is the partition function $\chi(\beta, N)$ for N such oscillators?

   (c) What is the classical grand partition function $X(\beta, \mu)$?

   (d) Compute the mean value of N by differentiating the grand potential,

$$N = -\left(\frac{\partial J}{\partial \mu}\right)_T.$$

   Express J in terms of N.

   (e) Give an expression for the chemical potential μ as a function of N and T.

(f) Calculate the entropy from

$$S = -\left(\frac{\partial J}{\partial T}\right)_\mu ,$$

and express S in terms of N and T.

(g) Show that you can obtain the same result from the canonical ensemble, where

$$S = \frac{U - F}{T},$$

where

$$U = -\frac{\partial}{\partial \beta} \ln \chi(\beta, N),$$

and ($N \to \infty$)

$$F = -kT \ln \frac{\chi(\beta, N)}{N!}.$$

(h) Now proceed *quantum mechanically.* Calculate $\chi(\beta, N)$ for a system of quantum oscillators.

(i) Calculate U by differentiating $\ln \chi(\beta, N)$.

(j) Calculate $X(\beta, \mu)$ for such quantum oscillators.

(k) Calculate N by differentiating $\ln X(\beta, \mu)$, and solve for μ in terms of N and T.

(l) Calculate the energy from

$$U - \mu N = -\frac{\partial}{\partial \beta} \ln X(\beta, \mu),$$

and show that the result agrees with that found in part 1i.

2. This problem concerns Bose condensation in a nonrelativistic gas of spinless atoms governed by Bose-Einstein statistics.

(a) Obtain a formula expressing the number of atoms N in terms of the chemical potential μ, as an integral over the different possible energy states of the atom.

(b) Argue that because the number of atoms at zero energy cannot be negative, the chemical potential must be negative.

(c) Then obtain an inequality of the form

$$\frac{N}{V} \leq AT^b,$$

and determine the constants A and b. [Recall

$$\int_0^\infty dx \frac{x^{\nu-1}}{e^x - 1} = \Gamma(\nu)\zeta(\nu),$$

where $\zeta(\nu)$ is the Riemann zeta function.]

(d) Show that this inequality cannot be satisfied if T is low enough. What is the minimum temperature T_c at which this inequality holds?

(e) For lower temperatures, the expression derived in part 2a, with $\mu = 0$, only applies to the number of atoms N' in excited states; the balance, $N_0 = N - N'$, must occupy the ground state. Obtain a formula for N_0 in terms of N, T, and T_c for $T < T_c$.

(f) Recalling that $S = (U-F)/T$, obtain a formula for the entropy per atom for temperatures below T_c in terms of the specific volume, $v = V/N$, and the characteristic wavelength,

$$\lambda = \frac{h}{\sqrt{2\pi mkT}}.$$

3. This problem concerns an ideal nonrelativistic Fermi gas of spin-1/2 particles at low temperatures.

(a) Obtain formulas for the number of particles N, and the energy U, as integrals over the single particle energy levels, expressed in terms of the chemical potential μ.

(b) Evaluate these integrals at $T = 0$, where all states up to the Fermi energy ε_F are occupied, and all those of higher energy are unoccupied. What is the relation between μ and ε_F?

(c) What is the average energy per particle in terms of ε_F?

(d) Give a physical argument why the specific heat of a Fermi gas at low temperature must be linear in T.

(e) Consider the function

$$f_\nu(\zeta) = \frac{1}{\Gamma(\nu)} \int_0^\infty dx \frac{x^{\nu-1}}{\zeta^{-1} e^x + 1}.$$

Sommerfeld showed that when $\zeta = e^\xi$, $\xi \to \infty$,

$$f_\nu(e^\xi) \sim \frac{\xi^\nu}{\Gamma(\nu+1)} \left[1 + \nu(\nu-1)\frac{\pi^2}{6}\frac{1}{\xi^2} + \mathcal{O}\left(\frac{1}{\xi^4}\right) \right].$$

Then calculate N/V and U/V as functions of T and μ for small T.

(f) Use the constancy of N/V to express $\mu(T)$ in terms of $\mu(0) = \varepsilon_F$:

$$\mu(T) \approx \varepsilon_F \left[1 + A \left(\frac{kT}{\varepsilon_F} \right)^2 \right],$$

and determine the constant A.

(g) Then show that

$$\frac{U}{N} = \frac{3}{5}\varepsilon_F \left[1 + B \left(\frac{kT}{\varepsilon_F} \right)^2 \right],$$

and determine the constant B.

(h) What is the specific heat of the Fermi gas at low temperature? Compare with the estimate in part 3d.