Physics 5163. Homework 5
Due Wednesday, April 1, 2009

March 10, 2009

1. Suppose that we evaluate, approximately, Hankel's representation for the gamma function
$$\frac{1}{\Gamma(n)} = \frac{1}{2\pi i}\int_C \frac{e^z}{z^n} dz$$
by choosing C to be a path passing not through the saddle point n, but rather through another real positive point x_0. In the Gaussian approximation (that is, keeping only the quadratic term in the Taylor series) show that the result is valid providing that
$$\frac{|x_0 - n|}{n} = o(n^{-1/3}),$$
where the notation means
$$\lim_{n\to\infty} n^{1/3} o(n^{-1/3}) = 0.$$

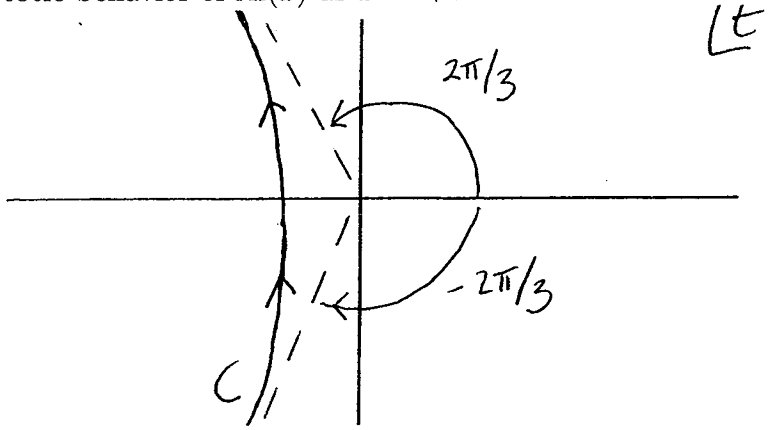
2. Using Hankel's representation for the Γ function and the saddle-point method given in class, compute the $O(1/n)$ correction in the asymptotic expansion of $\ln \Gamma(n)$.

3. The Airy function is defined by
$$\mathrm{Ai}(x) = \frac{1}{2\pi i}\int_C dt\, \exp\left(xt - \frac{1}{3}t^3\right),$$
where C is a contour stating at $\infty e^{-2\pi i/3}$ and ending at $\infty e^{2\pi i/3}$, as shown in the following figure.

(a) Determine the stationary points of the exponent.

(b) Which stationary point is a saddle point for which the path of steepest descents crosses the real axis in an imaginary direction?

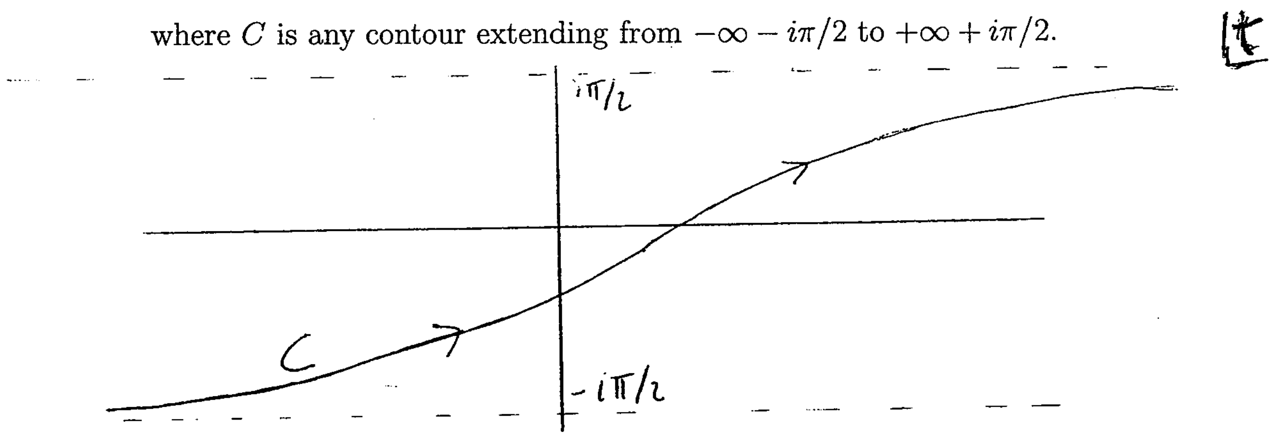
(c) Use the method of steepest descent to determine the leading asymptotic behavior of Ai(x) as $x \to +\infty$.



4. The Bessel function J_0 can be represented by the contour integral

$$J_0(x) = \mathrm{Re}\frac{1}{\pi i}\int_C dt\, e^{ix\cosh t},$$

where C is any contour extending from $-\infty - i\pi/2$ to $+\infty + i\pi/2$.



Find the leading behavior of $J_0(x)$ as $x \to \infty$ by the steepest-descents method. [Hint: the path of steepest descents makes a 45° angle with the real t axis.