Chapter 18

More about Fermions

Recall Stefan’s law for photons, Eq. (16.55), which gives the relation between the total energy and the temperature,

\[ U_\gamma = \frac{8\pi V}{(2\pi\hbar c)^3} (kT)^4 \Gamma(4) \zeta(4). \]  

We can similarly work out the corresponding relation for neutrinos. Because there is only one helicity state,

\[ U_\nu = \sum_l \frac{\varepsilon_l}{e^{\beta \varepsilon_l} + 1} = \frac{4\pi V}{(2\pi)^3 \hbar c} \int_0^\infty dk \frac{k^3}{e^{\beta \hbar ck} + 1} = \frac{4\pi V}{(2\pi \hbar c)^3} (kT)^4 \frac{7}{8} \Gamma(4) \zeta(4), \]

which uses the evaluation

\[
\int_0^\infty dx \frac{x^{n-1}}{e^x + 1} = \int_0^\infty dx x^{n-1} e^{-x} \sum_{k=0}^\infty (-1)^k e^{-kx} \\
= \sum_{k=0}^\infty (-1)^k \int_0^\infty dx x^{n-1} e^{-(k+1)x} = \sum_{k=0}^\infty (-1)^k \frac{\Gamma(n)}{(k+1)^n} \\
= \sum_{k=0}^\infty \frac{\Gamma(n)}{(k+1)^n} - 2 \sum_{l=1}^\infty \frac{\Gamma(n)}{(2l)^n} \\
= \zeta(n) \Gamma(n) \left( 1 - \frac{1}{2^{n-1}} \right). 
\]  

18.1 Temperature dependence of chemical potential

Let us define the Fermi distribution by

\[ f(\varepsilon) = \frac{1}{e^{\beta (\varepsilon - \mu)} + 1}. \]
The number of Fermions is

\[ N = \int_0^\infty d\varepsilon f(\varepsilon)g(\varepsilon) \equiv \int_0^\infty d\varepsilon N(\varepsilon), \]  

(18.5)
in terms of the density of states \( s = 1/2 \)

\[ g(\varepsilon) = \frac{4\pi(2m)^{3/2}}{(2\pi\hbar)^3}V\varepsilon^{1/2}. \]  

(18.6)

As we have seen in Eq. (17.35), at \( T = 0 \),

\[ N = \frac{2}{3}g(\varepsilon_F)\varepsilon_F, \]  

(18.7)

with \( \varepsilon_F = \mu(0) \), so

\[ g(\varepsilon_F) = \frac{3}{2}\frac{N}{\varepsilon_F}. \]  

(18.8)

As the temperature increases from zero, particles are removed below the point at which the distribution equals \( 1/2 \), \( \varepsilon = \mu \), and put above \( \mu \). Since the density of states increases with \( \varepsilon \), this must mean that \( \mu \) decreases. Let’s see this quantitatively as follows. Let \( g(\varepsilon) = \frac{d}{d\varepsilon}G(\varepsilon) \), where we choose \( G(0) = 0 \).

Then we can integrate by parts:

\[ N = \int_0^\infty d\varepsilon f(\varepsilon)\frac{dG(\varepsilon)}{d\varepsilon} = -\int_0^\infty d\varepsilon \frac{df(\varepsilon)}{d\varepsilon}G(\varepsilon). \]  

(18.9)

Now

\[-\frac{df(\varepsilon)}{d\varepsilon} = \frac{\beta e^{\beta(\varepsilon-\mu)}}{(e^{\beta(\varepsilon-\mu)}+1)^2} = \frac{\beta}{4\cosh^2 \frac{\beta(\varepsilon-\mu)}{2}}, \]  

(18.10)

which is even about \( \mu \). Further note that

\[-\int_{-\infty}^{\infty} d\varepsilon \frac{df(\varepsilon)}{d\varepsilon} = f(-\infty) - f(\infty) = 1, \]  

(18.11)

so, as \( \beta \to \infty \),

\[-\frac{df(\varepsilon)}{d\varepsilon} = \delta(\varepsilon - \mu), \]  

(18.12)

which is obvious, because \( f(\varepsilon) \) becomes a step function as \( \beta \to \infty \).

Now expand \( G(\varepsilon) \) around \( \mu \):

\[ G(\varepsilon) = G(\mu) + (\varepsilon - \mu)G'(\mu) + \frac{(\varepsilon - \mu)^2}{2}G''(\mu) + \ldots, \]  

(18.13)

where the first derivative term gives nothing in (18.9) because \( f' \) is even about \( \mu \). The second derivative term gives

\[-\int_{0}^{\infty} d\varepsilon \frac{(\varepsilon - \mu)^2}{2} \frac{df(\varepsilon)}{d\varepsilon} = -\frac{1}{2\beta^2} \int_{-\infty}^{\infty} dx x^2 \frac{df(x)}{dx}. \]
18.2. SPECIFIC HEAT AT LOW TEMPERATURE

\[
-\frac{1}{\beta^2} \int_0^\infty x^2 \frac{df(x)}{dx} = \frac{2}{\beta^2} \int_0^\infty x f(x) dx \\
= \frac{2}{\beta^2} \int_0^\infty dx \frac{x}{e^x + 1} = \frac{2}{\beta^2} \frac{1}{2} \Gamma(2) \zeta(2) \\
= \frac{1}{\beta^2} \frac{\pi^2}{6},
\]

(18.14)

where in the first step we defined \( x = \beta (\varepsilon - \mu) \) and regarded \( \beta \mu \) as infinitely large and negative, and in the penultimate step we used Eq. (18.3). Thus, we conclude that

\[
N = G(\mu) + \frac{\pi^2}{6} (kT)^2 G''(\mu) + \ldots
\]

(18.15)

Now

\[
G(\varepsilon) = \frac{2}{3} \varepsilon^{3/2} A, \\
G''(\varepsilon) = \frac{1}{2} \varepsilon^{-1/2} A,
\]

(18.16, 18.17)

with

\[
A = \frac{4\pi(2m)^{3/2}}{(2\pi \hbar)^3} V,
\]

(18.18)

and so

\[
N = A \left[ \frac{2}{3} \mu^{3/2} + \frac{\pi^2}{12} (kT)^2 \mu^{-1/2} + \ldots \right] = A \frac{2}{3} \varepsilon_F^{3/2},
\]

(18.19)

since the number of particles cannot change as the temperature changes. Therefore,

\[
\mu(T) = \varepsilon_F \left[ 1 - \frac{\pi^2}{12} \left( \frac{kT}{\varepsilon_F} \right)^2 + \ldots \right].
\]

(18.20)

18.2 Specific heat at low temperature

The internal energy can be written as

\[
U = \int_0^\infty d\varepsilon f(\varepsilon) g(\varepsilon) \varepsilon.
\]

(18.21)

Now let \( g(\varepsilon) \varepsilon = dG/d\varepsilon \), so again by integrating by parts

\[
U = -\int_0^\infty d\varepsilon \frac{df}{d\varepsilon} G(\varepsilon).
\]

(18.22)

Once again, expand \( G \) about \( \mu \), so by the same arguments

\[
U = G(\mu) + \frac{\pi^2}{6} (kT)^2 G''(\mu) + \ldots
\]

(18.23)
Here
\[
G(\varepsilon) = \frac{2}{5} A\varepsilon^{5/2}, \quad (18.24)
\]
\[
G''(\varepsilon) = \frac{3}{2} A\varepsilon^{1/2}, \quad (18.25)
\]
so
\[
U = \frac{2}{5} A\mu^{5/2} \left[ 1 + \frac{5\pi^2}{8} \left( \frac{kT}{\mu} \right)^2 + \ldots \right]. \quad (18.26)
\]
Inserting the temperature dependence for the chemical potential, Eq. (18.20),
\[
U = \frac{2}{5} A\varepsilon_F^{5/2} \left[ 1 - \frac{5\pi^2}{24} \left( \frac{kT}{\varepsilon_F} \right) + \ldots \right] \left[ 1 + \frac{5\pi^2}{8} \left( \frac{kT}{\mu} \right)^2 + \ldots \right]
\]
\[
= \frac{2}{5} A\varepsilon_F^{5/2} \left[ 1 + \frac{5\pi^2}{12} \left( \frac{kT}{\mu} \right)^2 + \ldots \right]. \quad (18.27)
\]
Then the specific heat is
\[
c_v = \frac{\partial U}{\partial T} = \frac{\pi^2}{3} A\varepsilon_F^{1/2} k^2 T. \quad (18.28)
\]
Write this in terms of the density of states at the Fermi surface,
\[
g(\varepsilon_F) = A\varepsilon_F^{1/2}; \quad (18.29)
\]
\[
c_v = \gamma T, \quad \gamma = \frac{\pi^2}{3} k^2 g(\varepsilon_F), \quad (18.30)
\]
or in view of Eq. (18.7),
\[
c_v = \frac{\pi^2 kT}{2 \varepsilon_F}. \quad (18.31)
\]
This result is so simple, there must be a more direct method to obtain it.
Recall,
\[
S = \frac{1}{T} (U - \mu N) + k \ln \chi, \quad (18.32)
\]
or in terms of discrete energy levels, and the notation
\[
f_j = \frac{1}{e^{\beta(\varepsilon_j - \mu)} + 1}, \quad (18.33)
\]
with \( \epsilon_j = \varepsilon_j - \mu, \)
\[
S = k\beta \sum_j f_j (\varepsilon_j - \mu) + k \sum_j \ln \left( 1 + e^{-\beta(\varepsilon_j - \mu)} \right)
\]
\[
= k\beta \sum_j f_j \varepsilon_j + k \sum_j \left( 1 + e^{-\beta\varepsilon_j} \right). \quad (18.34)
\]
Now for the specific heat at constant volume \((\delta W = 0)\)

\[
c_v = \left(\frac{\partial U}{\partial T}\right)_V \equiv \frac{\delta Q}{dT} = T \frac{dS}{dT} = -\beta \frac{dS}{d\beta}
\]

\[
= -k\beta \sum_j \left[ \frac{df_j}{d\beta} \beta \epsilon_j + f_j \frac{d}{d\beta} (\beta \epsilon_j) - \frac{e^{-\beta \epsilon_j}}{e^{-\beta \epsilon_j} + 1} \frac{d}{d\beta} (\beta \epsilon_j) \right]. \tag{18.35}
\]

The last two terms cancel leaving us with

\[
c_v = -k\beta^2 \sum_j \frac{df_j}{d\beta} \beta \epsilon_j = \sum_j \epsilon_j \frac{df_j}{dT}, \tag{18.36}
\]

which might appear obvious, but it is not. The derivative appearing here is

\[
\frac{df_j}{d\beta} = \frac{d}{d\beta} \frac{1}{e^{\beta \epsilon_j} + 1} = -\frac{e^{\beta \epsilon_j}}{(e^{\beta \epsilon_j} + 1)^2} \frac{d}{d\beta} (\beta \epsilon_j)
\]

\[
= -\frac{1}{e^{\beta \epsilon_j} + 1} \left( 1 - \frac{1}{e^{\beta \epsilon_j} + 1} \right) (\epsilon_j + \beta \frac{d\epsilon_j}{d\beta})
\]

\[
= -f_j (1 - f_j) \left( \epsilon_j + \beta \frac{d\epsilon_j}{d\beta} \right), \tag{18.37}
\]

or

\[
c_v = k\beta^2 \sum_j f_j (1 - f_j) \left( \epsilon_j^2 + \beta \epsilon_j \frac{d\epsilon_j}{d\beta} \right). \tag{18.38}
\]

Only the region near the Fermi surface contributes to the specific heat:

\[
f_j (1 - f_j) = \frac{1}{e^{\beta \epsilon_j} + 1} \frac{1}{e^{-\beta \epsilon_j} + 1} = -\frac{1}{\beta \frac{d\epsilon_j}{d\beta}} \rightarrow kT \delta(\epsilon), \tag{18.39}
\]

as \(T \rightarrow 0\) [see Eq. (18.12)].

Now put in the density of states:

\[
c_v = k\beta^2 g(\epsilon_F) \int_{-\infty}^{\infty} \epsilon e^2 f(1 - f)
\]

\[
= k g(\epsilon_F) \int_{-\infty}^{\infty} dx x^2 f(x)(1 - f(x)), \tag{18.40}
\]

where in the second line we set \(x = \beta \epsilon\). Note that in the scaled variable \(x\), since we are considering the limit \(\beta \rightarrow \infty\), we cannot regard \(f(x)(1 - f(x))\) as a \(\delta\) function, but we have already evaluated the latter integral at low temperature,

\[
- \int_{-\infty}^{\infty} dx x^2 \frac{df}{dx} = \frac{\pi^2}{3}. \tag{18.41}
\]

in Eq. (18.14). Thus,

\[
c_v = k^2 T g(\epsilon_F) \frac{\pi^2}{3}, \tag{18.42}
\]
which is the result (18.30). Here we neglected the term involving

$$\beta c \frac{de}{d\beta} = -\beta c \frac{d\mu}{d\beta} = cT \frac{d\mu}{dT} \approx 2c(\mu(T) - \varepsilon_r), \quad (18.43)$$

using Eq. (18.20), since the latter is an odd function of $\epsilon$, while $f(1-f)$ is even.