Chapter 5

Decay and Scattering

5.1 Two-Body Decay

First we consider the 2-body kinematics of a relativistic decay process, such as that of a pion into a muon and a neutrino:

$$\pi^+ \to \mu^+ \nu_\mu. \tag{5.1}$$

We suppose that the decaying particle has mass M and the two decay products have masses m_1 and m_2. We first consider the frame in which the decaying particle is at rest.

Momentum conservation says

$$0 = \mathbf{p}_1 + \mathbf{p}_2, \tag{5.2}$$

while energy conservation states

$$Mc^2 = E_1 + E_2, \quad E_a = \sqrt{p_a^2 c^2 + m_a^2 c^4}. \tag{5.3}$$

Squaring the latter equation,

$$(Mc^2)^2 = E_1^2 + E_2^2 + 2E_1 E_2, \tag{5.4}$$

and rearranging and squaring again, we get

$$(E_1^2 - E_2^2)^2 - 2(Mc^2)^2(E_1^2 + E_2^2) + (Mc^2)^4. \tag{5.5}$$

Substituting in

$$E_1^2 = p^2 c^2 + m_1^2 c^4, \quad E_2^2 = p^2 c^2 + m_2^2 c^4, \tag{5.6}$$

we obtain

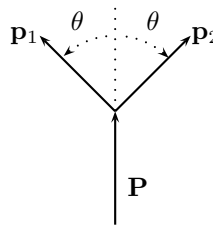$$p^2 = \frac{c^2}{4M^2} \left[M^2 - (m_1 + m_2)^2 \right] \left[M^2 - (m_1 - m_2)^2 \right]. \tag{5.7}$$

Figure 5.1: Decay of a particle with momentum **P** into two equal-mass particles, moving symmetrically with respect to **P**.

Note that this requires that $M > m_1 + m_2$, otherwise this decay channel is closed.

Some special cases of this general formula are noteworthy. If the two daughter particles are of equal mass, $m_1 = m_2 = m$, the formula for the momentum of each is

$$p = \frac{Mc}{2}\sqrt{1 - \frac{4m^2}{M^2}}. \tag{5.8}$$

And if one of the masses is zero, say $m_2 = 0$, and $m_1 = m$, we have

$$p = \frac{Mc}{2}\left(1 - \frac{m^2}{M^2}\right) \tag{5.9}$$

How does the same process look in the frame where the original particle is moving with velocity **v**, say along the z axis, perpendicular to the original decay axis? Its z component of momentum is then

$$P = M\gamma v = Mc\gamma\beta, \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{5.10}$$

For simplicity, let's consider the case of equal mass daughters, and in which, in the rest frame, the daughters move along the x axis. The decay process is then as shown in Fig. 5.1. Due to these simplifying assumptions, the two daughter momenta have equal magnitude, and make equal and opposite angles with respect to the z axis,

$$p = p_1 = p_2, \quad \theta_1 = -\theta_2 = \theta. \tag{5.11}$$

Momentum in the z direction is conserved,

$$P = 2p_1 \cos\theta. \tag{5.12}$$

Energy is also conserved:

$$2E_1 = E = Mc^2\gamma, \tag{5.13}$$

or

$$4(p^2c^2 + m^2c^4) = P^2c^2 + M^2c^4. \tag{5.14}$$

This tells us that

$$p = \frac{1}{2}\sqrt{P^2 + (M^2 - 4m^2)c^2}, \tag{5.15}$$

and from Eq. (5.12),

$$\cos\theta = \sqrt{\frac{1}{1 + \frac{(M^2 - 4m^2)c^2}{P^2}}}. \tag{5.16}$$

We can verify these results by performing a Lorentz transformation. Because the momentum of a particle is a four-vector, $p^\mu = (E/c, \mathbf{p})$, and in the rest frame of the decaying particle, the momentum of one of the daughters is

$$p^\mu = \left(\frac{Mc}{2}, \frac{Mc}{2}\sqrt{1 - \frac{4m^2}{M^2}}, 0, 0\right). \tag{5.17}$$

Now we make a Lorentz transformation (boost) in the z direction with velocity v:

$$p_1^{0\prime} = \gamma(p_1^0 + \beta p_1^z) = \gamma p_1^0 = \gamma\frac{Mc}{2}, \tag{5.18a}$$

$$p_1^{z\prime} = \gamma(p_1^z + \beta p_1^0) = \gamma\beta\frac{Mc}{2} = \frac{1}{2}P, \tag{5.18b}$$

$$p_1^{x\prime} = p_1^x = \frac{Mc}{2}\sqrt{1 - \frac{4m^2}{M^2}}. \tag{5.18c}$$

Indeed,

$$\frac{p_1^{x\prime}}{p_1^{z\prime}} = \tan\theta = \frac{Mc}{P}\sqrt{1 - \frac{4m^2}{M^2}}, \tag{5.19}$$

is equivalent to Eq. (5.16). For more general situations see Problem 1.

5.2 Nonrelativistic scattering

Let us consider the scattering of a particle by a central potential, as sketched in Fig. 5.2. We work in relative coordinates, where the reduced mass is $\mu = m_1 m_2/(m_1 + m_2)$. The central potential is $V(r)$. At infinity, the particle kinetic energy is $E = \frac{1}{2}\mu v_\infty^2$. The energy, of course, is conserved, as is the angular momentum, which at infinity is $L_z = \mu b v_\infty$. The angle through which the particle turns is given by the indefinite integral [see Eq. (2.31)],

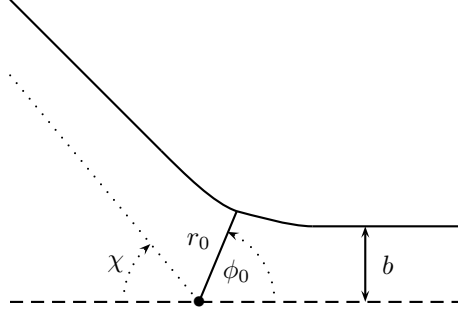$$\phi = \int \frac{L_z dr/r^2}{\sqrt{2\mu(E - V(r)) - L_z^2/r^2}}. \tag{5.20}$$

Figure 5.2: Scattering by a central potential. The particle comes from infinity with an impact parameter b, and is scattered through an angle χ. The distance of closest approach, r_0 occurs where the particle has moved through an angle ϕ_0 relative to the center.

The angle at closest approach is

$$\phi_0 = \int_{r_0}^{\infty} \frac{L_z dr/r^2}{\sqrt{2\mu(E - V(r)) - L_z^2/r^2}}$$

$$= \int_{r_0}^{\infty} \frac{\frac{b}{r^2} dr}{\sqrt{1 - \frac{b^2}{r^2} - \frac{2V}{\mu v_\infty^2}}}. \tag{5.21}$$

Because of the symmetry of the orbit about the midpoint, the scattering angle is related to ϕ_0 by

$$\chi = |\pi - 2\phi_0|, \tag{5.22}$$

where the absolute value appears because $2\phi_0$ might be bigger than π, for example, with an attractive potential.

Equation (5.21) determines the scattering angle in terms of the impact parameter, $\chi(b)$. We will assume here this is a monotone function, so it may be inverted, to give $b(\chi)$. (It fails to be so, for example, for scattering of magnetic monopoles, but this can be dealt with by summing over the various branches.) The way scattering is described is in terms of a scattering cross section,

$$d\sigma = \frac{dN}{n}, \tag{5.23}$$

where dN is the number of particles per unit time scattered through angles between χ and $\chi + d\chi$, and n is the number of particles incident per unit time per unit area. Thus, $d\sigma$ is an area, which is here the area of the annulus between

radii b and $b + db$:

$$d\sigma = 2\pi b\, db = 2\pi b \left| \frac{db}{d\chi} \right| d\chi, \tag{5.24}$$

where the absolute value sign refers to the fact that usually χ decreases as b increases. It is usual to express the differential cross section in terms of the solid angle subtended,

$$d\Omega = 2\pi \sin \chi\, d\chi, \tag{5.25}$$

so

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin \chi} \left| \frac{db}{d\chi} \right|. \tag{5.26}$$

Now we specialize to the Coulomb potential,

$$V(r) = -\frac{\alpha}{r}. \tag{5.27}$$

Then our equation for the central angle is, making the usual substitution $1/r = s$,

$$\phi_0 = \int_0^{1/r_0} \frac{ds}{\sqrt{\frac{1}{b^2} + \frac{\alpha^2}{\mu^2 b^4 v_\infty^4} - \left(s - \frac{\alpha}{\mu b^2 v_\infty^2} \right)^2}}. \tag{5.28}$$

The indefinite integral of this is

$$\phi = \tilde{\phi} + \arcsin \frac{bs - \xi}{\sqrt{1 + \xi^2}}, \quad \xi = \frac{\alpha}{\mu v_\infty^2 b}, \quad \tilde{\phi} = \arcsin \frac{\xi}{\sqrt{1 + \xi^2}}. \tag{5.29}$$

That is, the orbit is determined by

$$\frac{b}{r} = \xi + \sqrt{1 + \xi^2} \sin(\phi - \tilde{\phi}). \tag{5.30}$$

This result is the same as Eq. (2.34), of course. The minimum distance r_0 is achieved when $\phi - \tilde{\phi} = \frac{\pi}{2}$; therefore,

$$\cos \phi_0 = \cos(\pi/2 + \tilde{\phi}) = -\sin \tilde{\phi} = -\frac{\xi}{\sqrt{1 + \xi^2}} \tag{5.31}$$

This, in turn, implies that $\xi^2 = \cot^2 \phi_0$. Thus, the impact parameter is related to the central angle by

$$b^2 = \frac{\alpha^2}{\mu^2 v_\infty^4} \tan^2 \phi_0, \tag{5.32}$$

which says that

$$|b\, db| = \frac{\alpha^2}{\mu^2 v_\infty^4} \sin \phi_0 \sec^3 \phi_0\, d\phi_0. \tag{5.33}$$

Finally, substituting $\phi_0 = \frac{1}{2}(\pi - \chi)$, we obtain from Eq. (5.26) the famous Rutherford scattering formula

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{(2\mu v_\infty^2)^2} \frac{1}{\sin^4 \chi/2}. \tag{5.34}$$

5.3   Problems for Chapter 5

1. Of course, a moving particle need not decay in the symmetrical manner shown in Fig. 5.1. Solve the equations of energy and momentum conservation in the case that one daughter has magnitude of momentum p_1, with the momentum making an angle θ_1 with respect to the velocity of the decaying particle, and the second daughter having momentum magnitude p_2, with the momentum making an angle θ_2 with respect to the direction of **P**. Show that there is one undertermined quantity, say θ_2. What can you say about the distribution of daughter particle energies? It is not trivial to obtain an explicit expression; it may be most efficient to obtain the result by performing a Lorentz transformation. If the Lorentz transformation from the rest frame of the decaying particle is in a direction making an angle θ with respect to the rest-frame decay axis, obtain formulas giving θ_1, θ_2, E_1, and E_2, where $E_{1,2}$ are the energies of the daughter particles in the lab frame, in terms of θ, P the momentum of the decaying particle in the lab frame, and the particle masses.