Chapter 2

Generalized Coordinates. The Kepler-Coulomb Problem

2.1 Generalized coordinates

In general let us suppose we describe a system with generalized coordinates, $\{q_a\}_{a=1}^N$, and generalized velocities, $\{\dot{q}_a\}_{a=1}^N$; for example, they might be the Cartesian coordinates and velocities of $N/3$ particles moving in 3 dimensions, or the spherical polar coordinates and velocities of a particle: $\{q_a\} = \{r, \theta, \phi\}$, $\{\dot{q}_a\} = \{\dot{r}, \dot{\theta}, \dot{\phi}\}$. Let us suppose (although this may not always be true) that the Lagrangian depends only on q_a and $\dot{q}_a$, as well as, perhaps, on the time explicitly,

$$W_{12} = \int_{t_2}^{t_1} dt\, L(\{q_a\}, \{\dot{q}_a\}, t). \tag{2.1}$$

The action principle states

$$\begin{aligned}\delta W_{12} &= G_1 - G_2 \\ &= \int_{t_2}^{t_1} dt \left[\sum_a \left(\delta q_a \frac{\partial L}{\partial q_a} + \delta \dot{q}_a \frac{\partial L}{\partial \dot{q}_a}\right) + \delta t \frac{\partial L}{\partial t} + \frac{d\delta t}{dt}\left(L - \sum_a \dot{q}_a \frac{\partial L}{\partial \dot{q}_a}\right)\right],\end{aligned} \tag{2.2}$$

where the last term arises from the change in the time coordinate; see, for example, Eq. (1.85). Because $\delta \dot{q}_a = \frac{d}{dt}\delta q_a$, this variation may be rewritten as

$$\begin{aligned}\delta W_{12} &= \int_{t_2}^{t_1} dt \Bigg\{ \sum_a \left[\frac{d}{dt}\left(\delta q_a \frac{\partial L}{\partial \dot{q}_a}\right) - \delta q_a \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} - \frac{\partial L}{\partial q_a}\right)\right] \\ &\quad + \frac{d}{dt}\left[\delta t \left(L - \sum_a \dot{q}_a \frac{\partial L}{\partial \dot{q}_a}\right)\right] + \delta t \left[\frac{\partial L}{\partial t} - \frac{d}{dt}\left[L - \sum_a \dot{q}_a \frac{\partial L}{\partial \dot{q}_a}\right]\right]\Bigg\}.\end{aligned}$$

(2.3)

Defining the generalized momentum by

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \tag{2.4}$$

and the energy by

$$E = \sum_a \dot{q}_a p_a - L, \tag{2.5}$$

we have

$$\begin{aligned}\delta W_{12} &= \left[\sum_a \delta q_a p_a - \delta t E\right]_2^1 \\ &\quad - \int_{t_2}^{t_1} dt \left[\sum_a \delta q_a \left(\dot{p}_a - \frac{\partial L}{\partial q_a}\right) - \delta t \left(\frac{d}{dt}E + \frac{\partial L}{\partial t}\right)\right].\end{aligned} \tag{2.6}$$

By the action principle, we read off the generators and the equations of motion:

$$G = \sum_a p_a \delta q_a - E\delta t, \tag{2.7}$$

$$\dot{p}_a = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} = \frac{\partial L}{\partial q_a}, \quad \frac{d}{dt}E = -\frac{\partial L}{\partial t} = \frac{\partial}{\partial t}E. \tag{2.8}$$

2.2 Hamiltonian

The Hamiltonian is defined by the Legendre transformation:

$$H(\{q_a, p_a\}) = \sum_a \dot{q}_a p_a - L(\{q_a, \dot{q}_a\}), \tag{2.9}$$

which coincides, numerically, with the energy (2.5). However, the point here is that the variables are changed from $\{q_a, \dot{q}_a\}$ to $\{q_a, p_a\}$. Here the generalized canonical momentum is obtained from the Lagrangian by Eq. (2.4). Therefore,

$$\frac{\partial H}{\partial \dot{q}_a} = p_a - \frac{\partial L}{\partial \dot{q}_a} = 0. \tag{2.10}$$

The equations of motion for q_a and p_a are

$$\frac{\partial H}{\partial p_a} = \dot{q}_a, \quad \frac{\partial H}{\partial q_a} = -\frac{\partial L}{\partial q_a} = -\dot{p}_a, \tag{2.11}$$

where the last follows from the Euler-Lagrange equations (2.8). The Hamiltonian is time dependent only if explicitly so:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_a \left(\dot{q}_a \frac{\partial H}{\partial q_a} + \dot{p}_a \frac{\partial H}{\partial p_a}\right), \tag{2.12}$$

where the summand vanishes by virtue of the equation of motion (2.11). Thus, the complete set of Hamilton's equations are:

$$\dot{q}_a = \frac{\partial H}{\partial p_a}, \quad \dot{p}_a = -\frac{\partial H}{\partial q_a}, \quad \frac{dH}{dt} = \frac{\partial H}{\partial t}. \tag{2.13}$$

As sketched in Chapter 1, these results can be directly obtained from the action principle, which we rewrite as

$$W_{12} = \int_2^1 dt\, L = \int_2^1 dt \left(\sum_a p_a \dot{q}_a - H \right), \tag{2.14}$$

which has as its variation

$$\begin{aligned} \delta W_{12} &= \int_2^1 dt \left[\sum_a \left(\delta p_a \dot{q}_a + p_a \frac{d}{dt} \delta q_a - \delta q_a \frac{\partial H}{\partial q_a} - \delta p_a \frac{\partial H}{\partial p_a} \right) - \delta t \frac{\partial H}{\partial t} - \frac{d\delta t}{dt} H \right] \\ &= \int_2^1 \left[\frac{d}{dt} \left(\sum_a \delta q_a p_a - H\delta t \right) + \delta q_a \left(-\dot{p}_a - \frac{\partial H}{\partial q_a} \right) + \delta p_a \left(\dot{q}_a - \frac{\partial H}{\partial p_a} \right) \right. \\ &\qquad \left. + \delta t \left(\frac{dH}{dt} - \frac{\partial H}{\partial t} \right) \right]. \end{aligned} \tag{2.15}$$

The action principle (1.3) yields the Hamilton equations (2.13) as well as the expected form of the generator,

$$G = \sum_a p_a \delta q_a - H\delta t. \tag{2.16}$$

2.3 Motion of a particle in a $1/r$ potential

For a particle described in spherical polar coordinates, r, θ, ϕ, which are related to the Cartesian coordinates by

$$x = r \sin\theta \cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta, \tag{2.17}$$

the velocity vector is given in terms of the generalized coordinates and velocities by

$$\begin{aligned} \dot{\mathbf{r}} &= \hat{\mathbf{x}}(\dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi) \\ &\quad + \hat{\mathbf{y}}(\dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi) \\ &\quad + \hat{\mathbf{z}}(\dot{r}\cos\theta - r\dot{\theta}\sin\theta). \end{aligned} \tag{2.18}$$

When this is squared, there is considerable cancellation:

$$\dot{\mathbf{r}}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2, \tag{2.19}$$

which is reflective of the geometrical line element,

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{2.20}$$

We will consider the Lagrangian of a single particle in a central potential,

$$L = \frac{1}{2} m \dot{\mathbf{r}}^2 - V(r), \tag{2.21}$$

and, in fact, we will specialize to the Newtonian or Coulomb potential,

$$V(r) = -\frac{\alpha}{r}, \tag{2.22}$$

where in the Newtonian case of a light planet of mass m orbiting a very heavy sun of mass M, $\alpha = GMm$ in terms of Newton's constant G, while for the interaction between a heavy nucleus, of electric charge $+Ze$ and a light electron of charge $-e$, $\alpha = Ze^2$ (Gaussian units). The mathematics is identical in the two cases.

The generalized momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2\theta \dot{\phi}. \tag{2.23}$$

The equations of motion are

$$\dot{p}_r = m\ddot{r} = \frac{\partial L}{\partial r} = -\frac{\partial V}{\partial r} + mr\dot{\theta}^2 + mr\sin^2\theta\dot{\phi}^2, \tag{2.24a}$$

$$\dot{p}_\theta = \frac{\partial L}{\partial \theta} = mr^2 \sin\theta\cos\theta\dot{\phi}^2, \tag{2.24b}$$

$$\dot{p}_\phi = \frac{\partial L}{\partial \phi} = 0. \tag{2.24c}$$

The last equation is evidently a conservation law; in fact, it is easy to check that

$$p_\phi = L_z, \quad \mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{2.25}$$

so this is just the statement of the conservation of the z component of angular momentum. The remaining equations can be greatly simplified by recognizing that the motion is confined to a plane, which we may, without loss of generality, take to be the $z = 0$ plane, in which case $\theta = \pi/2$. This is consistent with the second equation of motion (2.24b), since both sides vanish in that case. The first, radial, equation (2.24a) then simplifies to

$$m\ddot{r} = -\alpha\frac{1}{r^2} + \frac{L_z^2}{mr^3}. \tag{2.26}$$

We can obtain a first integral of this differential equation by multiplying it by $\dot{r}$:

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + \frac{L_z^2}{2mr^2} - \frac{\alpha}{r}\right) = 0; \tag{2.27}$$

the quantity in parentheses is simply the energy, which we already know is conserved, since there is no explicit time dependence in this problem:

$$E = \sum_a p_a \dot{q}_a - L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{\alpha}{r} = \frac{m}{2}\left(\dot{r}^2 + \frac{L_z^2}{m^2r^2}\right) - \frac{\alpha}{r}. \tag{2.28}$$

The solution of the problem is now given by quadratures: That is, we solve for $\dot{r}$:

$$\dot{r} = \sqrt{\frac{2}{m}\left(E + \frac{\alpha}{r}\right) - \frac{L_z^2}{m^2 r^2}}, \tag{2.29}$$

or

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E + \frac{\alpha}{r}\right) - \frac{L_z^2}{m^2 r^2}}} \tag{2.30}$$

Since angular momentum is conserved, we can make this an equation for ϕ, because $d\phi = \frac{L_z}{mr^2}dt$:

$$\phi = \int \frac{L_z dr/r^2}{\sqrt{2m(E + \alpha/r) - L_z^2/r^2}}. \tag{2.31}$$

This solution in fact holds for a general central potential, with $\alpha/r \to V(r)$.

For the $1/r$ potential, the solution is simple. Let $s = 1/r$ and then we can write this as

$$\phi = \int ds \frac{1}{\sqrt{\frac{2m}{L_z^2}(E + \alpha s) - s^2}}, \tag{2.32}$$

where one can complete the square, and in terms of $s' = s - \alpha m/L_z^2$ find

$$\phi = \arcsin \frac{s'}{\sqrt{2mE/L_z^2 + \alpha^2 m^2/L_z^4}} + \text{constant}. \tag{2.33}$$

Therefore,

$$s = \frac{1}{r} = \frac{\alpha m}{L_z^2} + \frac{\alpha m}{L_z^2}\sqrt{1 + \frac{2EL_z^2}{m\alpha^2}}\sin(\phi - \phi_0). \tag{2.34}$$

ϕ_0 is an arbitrary constant; choose it to be $\phi_0 = -\pi/2$, so the above appears in standard form

$$\frac{p}{r} = 1 + e\cos\phi, \tag{2.35}$$

where

$$p = \frac{L_z^2}{\alpha m}, \quad e = \sqrt{1 + \frac{2EL_z^2}{m\alpha^2}}. \tag{2.36}$$

This is the equation of an ellipse, if $e < 1$, that is, for negative energies (bound states) $E < 0$. In that case, e is the eccentricity, and $2p$ is the latus rectum. The sun (or nucleus) lies at one of the foci of the ellipse. The distance of closest approach of the planet to the sun is called the perihelion. That distance occurs when $\phi = 0$, so

$$r_p = \frac{p}{1+e}; \tag{2.37}$$

the aphelion, the furthest distance from the sun occurs for $\phi = \pi$, or

$$r_a = \frac{p}{1-e}. \tag{2.38}$$

The semimajor axis is therefore

$$a = \frac{p}{1-e^2} \quad \text{so} \quad r_p = a(1-e); \tag{2.39}$$

a small calculation (homework) reveals the semiminor axis is

$$b = \frac{p}{\sqrt{1-e^2}}. \tag{2.40}$$

In physical terms, then,

$$a = -\frac{\alpha}{2E}, \quad E < 0, \quad \text{or} \quad E = -\frac{\alpha}{2a}. \tag{2.41}$$

What is the period of this closed orbit? The constancy of the angular momentum implies that of the areal velocity. Evidently, for a small angular change, the area swept out by the radius vector is

$$dS = \frac{1}{2} r^2 d\phi, \tag{2.42}$$

so

$$\frac{dS}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{L_z}{2m}. \tag{2.43}$$

This is Kepler's second law. Therefore,

$$dt = \frac{2m}{L_z} dS, \tag{2.44}$$

so the period is

$$T = \frac{2m}{L_z} S, \tag{2.45}$$

where S is the area enclosed by the orbit, here $S = \pi ab$. Therefore, inserting the various quantities we have

$$T = \pi \sqrt{\frac{m}{2}} \frac{\alpha}{(-E)^{3/2}}. \tag{2.46}$$

Note that the angular momentum cancels out. But because $E = -\alpha/(2a)$, this can be written in the more familiar form

$$T^2 = 4\pi^2 \frac{ma^3}{\alpha} = \frac{4\pi^2}{GM} a^3, \tag{2.47}$$

which says that the square of the period is proportional to the cube of the semimajor axis, Kepler's third law. Note the crucial point, in the planetary case, the mass of the planet drops out.

The same orbit equation (2.35) applies for $E > 0$, or $e > 1$. Now the perihelion distance is given by

$$r_p = \frac{p}{e+1} = a(e-1), \tag{2.48}$$

where

$$a = \frac{p}{e^2-1} = \frac{\alpha}{2E}. \tag{2.49}$$

The orbit is a hyperbola (a parabola if $E = 0$), and the distance of the "comet" from the sun, say, becomes infinite when $\phi = \arccos \frac{-1}{e}$.

A similar orbit equation to (2.35) applies for a repulsive field, with $\alpha < 0$. Then the energy is necessarily positive, $e > 1$, and

$$\frac{p}{r} = -1 + e\cos\phi. \tag{2.50}$$

Again the orbit is a hyperbola. The perihelion distance is given by

$$r_p = \frac{p}{e-1}, \tag{2.51}$$

but now the focus is outside of the hyperbola. The asymptotes of the hyperbola occurs at $\phi = \arccos \frac{1}{e}$. See figures in Landau and Lifshitz, pp. 37 and 39.

2.4 Axial Vector

The fact that the orbit closes is a remarkable feature of the $1/r$ potentials. It does not close for other central potentials except r^2. This reflects a hidden symmetry of the Kepler problem. In addition to the vector **L**, which is perpendicular to the plane of the orbit, there is another vector, in the plane of the orbit,

$$\mathbf{A} = \mathbf{v}\times\mathbf{L} - \alpha\frac{\mathbf{r}}{r}, \tag{2.52}$$

which is conserved. Indeed,

$$\begin{aligned}\frac{d}{dt}\mathbf{A} &= \dot{\mathbf{v}}\times(\mathbf{r}\times m\mathbf{v}) - \alpha\frac{\mathbf{v}}{r} + \alpha\mathbf{r}\frac{\mathbf{v}\cdot\mathbf{r}}{r^3}\\ &= (\mathbf{r}\mathbf{v}-\mathbf{v}\mathbf{r})\cdot(m\dot{\mathbf{v}} + \alpha\frac{\mathbf{r}}{r^3}) = 0,\end{aligned} \tag{2.53}$$

by virtue of the equation of motion. Thus **A** is constant in direction and magnitude. At the perihelion it is easy to evaluate it, because there **v** and **r** are perpendicular. We see that **A** points along the semimajor axis from the sun to the perihelion point and has magnitude

$$A = mv^2 r_p - \alpha = r_p 2T_p - \alpha = 2r_p\left(-\frac{\alpha}{2a} + \frac{\alpha}{r_p}\right) - \alpha = \alpha\left(-\frac{r_p}{a} + 1\right) = \alpha e, \tag{2.54}$$

so

$$\mathbf{A} = \hat{\mathbf{r}}_p \alpha e. \tag{2.55}$$

Because it points in the direction of the axis of the ellipse, it might be dubbed the axial vector (although it is a "polar" vector!). It is often referred to as the Laplace-Runge-Lenz vector, or by some subset of those names, but it was actually first discovered by Hermann in 1710, and generalized by Bernoulli in the same year.

What is the corresponding symmetry? It turns out to be the 4-dimensional rotation group, $O(4)$, equivalent to $SU(2) \times SU(2)$, but since this is of most interest in quantum mechanics, we will not stop to elucidate this point now.

2.5 Reduced mass

In fact, the mass of the sun is not infinite, and it moves somewhat as a planet orbits around it. We have then a two-body problem, but with the potential depending only on the distance between the bodies. Let us state this problem generally, which is governed by the Lagrangian

$$L = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - V(\mathbf{r}_1 - \mathbf{r}_2). \tag{2.56}$$

Let us introduce the center of mass and relative coordinates,

$$M\mathbf{R} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \tag{2.57}$$

with $M = m_1 + m_2$. This means that

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r}, \quad \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}. \tag{2.58}$$

When this is substituted into the kinetic energy, the cross term between $\dot{\mathbf{r}} \cdot \dot{\mathbf{R}}$ cancels out, and we are left with

$$T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2, \tag{2.59}$$

where the reduced mass appears,

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \quad \text{or} \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{2.60}$$

Thus, the kinetic energy breaks up into the energy of the center of mass and that of the relative motion. The equations for the relative motion are the same as before, but with the mass m replaced by the reduced mass $\mu = mM/(m+M) \approx m$ if $m \ll M$.

The same kind of breakup occurs for the linear and angular momentum. For the former,

$$\mathbf{P} = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = M\dot{\mathbf{R}}, \tag{2.61}$$

as we already know, see Eq. (1.67). The angular momentum breaks up like the energy,

$$\mathbf{L} = \mathbf{R} \times M\dot{\mathbf{R}} + \mathbf{r} \times \mu\dot{\mathbf{r}} = \mathbf{L}_{\text{CM}} + \mathbf{L}_{\text{R}}, \tag{2.62}$$

into the angular momentum of the center of mass, and that of the relative motion. So the modification entailed by the finite mass of the sun is trivial. This is not the case when the effect of forces due to third bodies, e.g. Jupiter, are considered!

2.6 Problems for Chapter 2

1. Prove that $p_\phi = L_z$, Eq. (2.25)

2. Prove that the semiminor axis of the elliptical orbit is given by Eq. (2.40).

3. Prove the statements made about the hyperbolic orbits in the last two paragraphs of Sec. 2.3.