Final Examination
Physics 5013, Mathematical Methods of Physics

December 16, 2011

Instructions: Attempt all parts of this exam. If you get stuck on one part, assume an answer and proceed on. Do not hesitate to ask questions.

Remember this is a closed book, closed notes, exam. *Good luck! and have a wonderful holiday!*

1. (a) For $\operatorname{Re} s > 1$ evaluate

$$F(s) = \sum_{n=1}^{\infty} (-1)^n n^{-s}$$

as proportional to the Riemann zeta function. Justify your operations by referring to the theory of absolute convergence. [Hint: Think about the n even and n odd terms separately.]

(b) What is $F(0)$? Evaluate this divergent sum using the generic summation method, for example. By comparing the result with the formula in part 1a evaluate the Riemann zeta function at zero, $\zeta(0)$.

(c) The Riemann zeta function $\zeta(s)$ has a pole at $s = 1$. By using the formula in part 1a for $s = 1 + \epsilon$, where ϵ is small, show

$$\zeta(1 + \epsilon) = \frac{1}{\epsilon} + A + \mathcal{O}(\epsilon),$$

and determine the constant A. Hint: you should encounter the sum

$$\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n} = \ln 2 \left(\gamma - \frac{1}{2} \ln 2 \right),$$

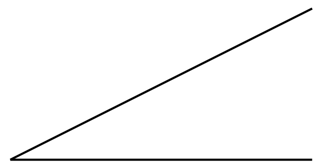

Figure 1: A perfectly conducting wedge of opening angle α. The two lines represent planes indefinitely extended perpendicular to the plane of the page.

where γ is Euler's constant.

2. Consider

$$f(z) = \int_0^z \frac{dx}{x^2+1}.$$

   (a) Evaluate the integral by means of a trigonometric substitution.

   (b) For sufficiently small z obtain a power series expansion of $f(z)$ by expanding the integrand.

   (c) What is the radius of convergence of the series? Why is it this value?

3. In this problem we are considering the solution of Poisson's equation

$$-\nabla^2\phi = 4\pi\rho \tag{1}$$

in a wedge geometry as shown in the Fig. 1, where we are interested in the interior of the wedge. ϕ is subject to having specified values on the planes $\theta = 0$, $\theta = \alpha$ (inhomogeneous Dirichlet boundary conditions).

   (a) The corresponding Green's function satisfies

$$-\nabla^2 G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \tag{2}$$

   where G must vanish at $\theta = 0$ and $\theta = \alpha$. Express this in cylindrical coordinates, with origin on the apex of the wedge, so

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}.$$

Show that the Green's function can be written in the form

$$F(r,\theta,z;r',\theta',z') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(z-z')} N \sum_\nu \sin\nu\theta \sin\nu\theta' g(r,r'),$$

where the reduced Green's function satisfies

$$\left[-\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + k^2 + \frac{\nu^2}{r^2}\right] g(r,r') = \frac{1}{r}\delta(r-r').$$

Here ν is a discrete index (quantum number) that is specified by the boundary conditions on the wedge. Express ν as a multiple of an integer. What is the normalization constant N?

(b) Give a closed form for the reduced Green's function $g(r,r')$ in terms of modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$, which are independent solutions of

$$\left[-\frac{d^2}{dx^2} - \frac{1}{x}\frac{d}{dx} + 1 + \frac{\nu^2}{x^2}\right] \left\{ \begin{array}{c} I_\nu(x) \\ K_\nu(x) \end{array} \right\} = 0.$$

The Wronskian of these solutions is

$$I_\nu'(x) K_\nu(x) - K_\nu'(x) I_\nu(x) = \frac{1}{x}.$$

Note that $I_\nu(x)$ is finite as $x \to 0$ and $K_\nu(x)$ vanishes as $x \to \infty$.

(c) Combine Eqs. (1) and (2) to express $\phi(\mathbf{r})$ within the wedge in terms of the change density distributed throughout the volume of the wedge and the boundary values of ϕ on the wedge.

(d) Suppose the boundary values of the potential are zero, but there is a point charge on the line bisecting the wedge,

$$\rho(\mathbf{r}) = e\delta(z)\delta(r-r_0)\delta(\theta-\alpha/2).$$

Give an expression for the normal component of the electric field on the lower wedge boundary,

$$E_\theta(r,0,z) = -\frac{1}{r}\frac{\partial}{\partial\theta}\phi(r,\theta,z)\Big|_{\theta=0}.$$

(e) Integrate the normal component of the electric field over the lower conductor to determine the total charge induced thereon:

$$4\pi Q = \int_0^\infty dr \int_{-\infty}^\infty dz \, E_\theta(r, 0, z).$$

To evaluate the integrals, you will need the limiting value

$$\lim_{k\to 0} I_\nu(kx) K_\nu(ky) = \frac{1}{2\nu}\left(\frac{x}{y}\right)^\nu.$$

After using the result of Problem 2, you should obtain the answer $Q = -e/2$.