Final Examination
Physics 5013, Mathematical Methods of Physics

December 15, 2006

Instructions: Attempt all parts of this exam. If you get stuck on one part, assume an answer and proceed on. Do not hesitate to ask questions.

Remember this is a closed book, closed notes, exam. *Good luck! and have a wonderful holiday!*

1. Consider the region within a pair of intersecting perfectly conducting planes intersecting with dihedral angle Θ, as shown in the figure. The two planes are maintained at different potentials, zero and V_0, as shown. Find the electrostatic potential ϕ at all points between the planes as follows. Everything is expressed in terms of cylindrical coordinates (r, θ, z), where the origin is taken to lie on the intersection line between the two planes, the z axis, r is the distance from that axis, and θ is the angle around the axis measured from the lower plane. In these coordinates the Laplacian is

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.)$$

   (a) Using the Fourier decomposition of the delta function,

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(z-z')} = \delta(z - z'),$$

   argue that $(2\pi)^{-1/2} e^{ikz}$ is a "normalized" eigenfunction of the operator $\partial^2/\partial z^2$.

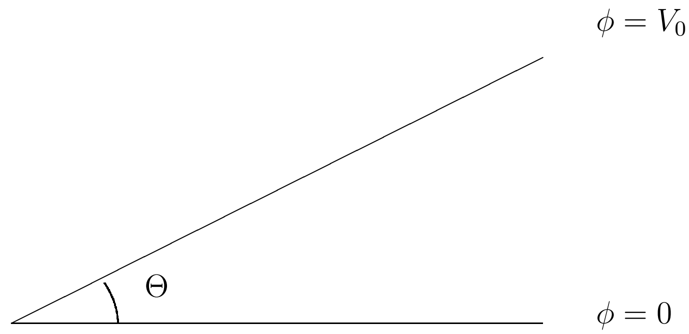

Figure 1: Two conducting planes intersecting at angle Θ. The planes extend indefinitely to the right, and have infinite extension into and out of the page. The lower plane is maintained at zero potential while the upper one is at potential V_0.

(b) Similarly, show that the eigenfunctions of $\partial^2/\partial\theta^2$ that vanish on the two planes are

$$A \sin \frac{m\pi\theta}{\Theta},$$

where m is a positive integer, and determine the normalization constant A. What are the statements of orthonormality and completeness for these eigenfunctions?

(c) Now write the Green's function for Laplace's equation, which satisfies

$$-\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \tag{1}$$

and vanishes on the planes, in terms of a reduced Green's function in the form

$$G(\mathbf{r}, \mathbf{r}') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(z-z')} \sum_{m=1}^{\infty} A^2 \sin \frac{m\pi\theta}{\Theta} \sin \frac{m\pi\theta'}{\Theta} g_m(r, r'; k),$$

and obtain the differential equation satisfied by g_m.

(d) Recalling that the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$ satisfy the differential equation

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - 1 - \frac{\nu^2}{x^2} \right) u(x) = 0,$$

solve the equation found in part 1c subject to the boundary condition that g_m vanish at $r = 0$ and $r = \infty$. In so doing you will need the Wronskian

$$K_\nu(x)I'_\nu(x) - I_\nu(x)K'_\nu(x) = \frac{1}{x},$$

where prime denotes differentiation. Recall that for $\nu > 0$, $I_\nu(x)$ vanishes at $x = 0$, while $K_\nu(x)$ diverges there; on the other hand, $K_\nu(x) \to 0$ as $x \to \infty$ while $I_\nu(x)$ diverges at infinity.

(e) Now use this Green's function to solve for the potential without charges within the wedge, which satisfies Laplace's equation,

$$-\nabla^2\phi = 0.$$

Combine this equation with that satisfied by the Green's function (1) to obtain a general formula for the potential at any interior point in terms of the potential on the bounding planes.

(f) For the case specified, where on the planes

$$\phi(\theta = 0) = 0, \quad \phi(\theta = \Theta) = V_0,$$

where V_0 is a constant, evaluate the formula found in the previous part, and obtain the simple expected result. [Hint: You will need to know the behavior of the modified Bessel functions for small argument:

$$I_\nu(x) \sim \frac{(x/2)^\nu}{\Gamma(\nu+1)}, \quad K_\nu(x) \sim \frac{\frac{1}{2}\Gamma(\nu)}{(x/2)^\nu},$$

as $x \to 0$. You may also need to construct the Fourier series of the linear function θ in terms of the functions $\sin m\pi\theta/\Theta$.]

2. In this problem we consider the summation of the divergent series

$$1 - 1 + 1 - 1 + 1 - 1 + \dots.$$

(a) Sum the series by any of the linear methods (Euler, Borel, or generic).

(b) Sum the same series by the zeta-function method by writing it as the difference of two series. By demanding that the series have the same value as in part 2a, determine in this way the value of the Riemann zeta function at zero, $\zeta(0)$.

3. Evaluate the definite integral

$$\int_0^\infty dx \frac{\sqrt{x}}{x^2 + a^2},$$

where a is a real constant. Do so by using a suitably defined contour integral and the residue theorem. State carefully where the branch line of the square root is chosen, and keep track of phases carefully.