First Examination
Physics 5013, Mathematical Methods of Physics

October 14, 2011

Instructions: Attempt all parts of this exam. If you get stuck on one part, assume an answer and proceed on. Do not hesitate to ask questions. Remember this is a closed book, closed notes, exam. *Good luck!*

1. (a) Use the Cauchy integral formula to prove that if a function f is analytic in a disk which is defined by a circular boundary and the interior thereof, the value of f at the center of the disk is equal to the mean of the values of the function on the circumference of the disk. That is, if the center of the disk is taken at the origin, and the radius of the disk is ρ,

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} d\theta f\left(\rho e^{i\theta}\right).$$

   (b) What does this say if $f(z) = e^z$? Give the resulting identity as an integral over a real function, depending on the parameter ρ.

2. Consider the real integral

$$f(y) = \int_0^y dx \frac{1}{\sqrt{1-x^2}}.$$

   (a) Evaluate this integral by a trigonometric substitution.

   (b) What is $f(1)$?

(c) Expand the square root in powers of x, and integrate term by term, and obtain a power-series expansion for $f(y)$:

$$f(y) = \sum_{n=0}^{\infty} a_n y^n.$$

What are the coefficients a_n?

(d) Use the ratio test to determine the radius of convergence of this series.

(e) Why is the radius of convergence what you determined in the previous part?

(f) Does the series converge at $y = 1$? Use Gauss' test, for example, to answer this question.

(g) Calculate the first three nonzero terms in the series evaluated at $y = 1$ to see how close it is to the value expected from part 2b. Is this a reasonable approximation?

3. Consider the closed contour integral

$$\oint_C dz \frac{1}{z^4 - 1}.$$

(a) If C is a contour that encloses all the singularities of the integrand, encircling them once in a positive sense, what is the value of the integral? Hint: use Cauchy's theorem to distort the contour to a sum of contributions from contour integrals each of which encircles exactly one singularity.

(b) What is the value of the integral if C only encircles the singularites that lie on the real axis?