Second Examination
Physics 5013, Mathematical Methods of Physics

November 20, 2006

Instructions: Attempt all parts of this exam. If you get stuck on one part, assume an answer and proceed on. Do not hesitate to ask questions. Remember this is a closed book, closed notes, exam. *Good luck!*

1. In this problem you are to evaluate the definite integral

$$I = \int_0^\infty dx \frac{\log x}{x^2 - 1},$$

using the residue theorem. Do this as follows:

   (a) Show that I may be obtained from a closed contour integral. Specify precisely what the integrand and the contour of integration is.

   (b) Evaluate the various parts of the contour integral. Use Jordan's lemma if appropriate.

   (c) Evaluate the entire contour integral by the residue theorem.

   (d) Putting this all together, give the value of I.

2. Consider the divergent series

$$S = 1 + 2 + 4 + 8 + \ldots = \sum_{n=0}^{\infty} 2^n.$$

   (a) Evaluate the series, if possible, using Euler summation.

   (b) Evaluate the series, if possible, using Borel summation.

(c) Evaluate the series, if possible, using the generic method based on the linearity properties.

3. The factorial function is given by the Euler formula

$$n! = \int_0^\infty dt\, t^n e^{-t}.$$

(a) Write the integrand in the form

$$e^{\phi(t)}$$

and find the stationary point.

(b) Determine the path of steepest descents passing through this point.

(c) Evaluate the leading asymptotic behavior of $n!$ for large n using the method of steepest descents. You should in this way recover the Stirling approximation.