First Examination
Physics 5013, Mathematical Methods of Physics

October 20, 2006

Instructions: Attempt all parts of this exam. If you get stuck on one part, assume an answer and proceed on. Do not hesitate to ask questions. Remember this is a closed book, closed notes, exam. *Good luck!*

1. In this problem you are to evaluate the definite integral

$$I = \int_0^\infty \frac{dx}{x} \frac{\sin x}{(x^2+a^2)^2}, \quad a > 0,$$

   using the residue theorem. Do this as follows:

   (a) Show that I is part of a closed contour integral. Specify precisely what the integrand and the contour of integration is.

   (b) Evaluate the various parts of the contour integral. Use Jordan's lemma if appropriate.

   (c) Evaluate the entire contour integral by the residue theorem.

   (d) Finally give the value of I as a function of a.

2. The tangent function may be expressed as a Taylor series about the origin:

$$\tan z = \sum_{n=0}^{\infty} a_n z^n.$$

   (a) Which coefficients a_n are zero? Why?

(b) Calculate the first three nonzero coefficients using any method you like.

(c) What is the radius of convergence of this power series? Why?

(d) Calculate the ratio of adjacent nonzero coefficients for large n in terms of this radius of convergence.

(e) How well is this asymptotic ratio satisfied for the first three nonzero coefficients that you calculated in part 2b? [It turns out that the ratio of the kth to $k+1$th nonzero coefficients agrees with the asymptotic limit to k decimal places!]