

Physics 5013. Homework 8
Due Wednesday, December 14, 2011

November 30, 2011

1. Suppose we have a second-order differential operator of the form

$$L = \frac{1}{f} \frac{d}{dx} \left(f \frac{d}{dx} \right) + q,$$

where f and q are functions of x . If y_1 and y_2 are independent solutions of

$$Ly = 0,$$

the Wronskian

$$\Delta(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

is different from zero. Prove that

$$\frac{d}{dx} \Delta = -\Delta \frac{d}{dx} \ln f,$$

and that

$$y_2(x) = \Delta(x_0) f(x_0) y_1(x) \int_{x_0}^x \frac{du}{f(u) y_1^2(u)},$$

where x_0 is a point at which

$$\begin{aligned} y_2(x_0) &= 0, & y_1(x_0) &\neq 0, \\ f(x_0) &\neq 0, & y_1'(x_0) &\neq 0. \end{aligned}$$

2. Recall that the Bessel functions of integer order are defined by

$$e^{(x/2)(z-1/z)} = \sum_{m=-\infty}^{\infty} z^m J_m(x),$$

or, with $x = kr$, $z = ie^{i\phi}$.

$$e^{ikr \cos \phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(kr).$$

Use this expression in the *two-dimensional* completeness statement for the functions

$$\frac{1}{2\pi} e^{i\mathbf{k}\cdot\mathbf{r}},$$

that is,

$$\int \frac{(d\mathbf{k})}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\mathbf{k}\cdot\mathbf{r}'} = \delta(\mathbf{r} - \mathbf{r}'),$$

where the right-hand side is a two-dimensional delta function, which in polar coordinates is

$$\delta(\mathbf{r} - \mathbf{r}') = \frac{1}{r} \delta(r - r') \delta(\theta - \theta'),$$

and $(d\mathbf{k})$ is the two-dimensional integration element, which is correspondingly given in polar coordinates as

$$(d\mathbf{k}) = k dk d\alpha.$$

In this way derive the completeness property of the Bessel functions,

$$\int_0^\infty k dk J_m(kr) J_m(kr') = \frac{1}{r} \delta(r - r').$$

3. Determine, directly, the one-dimensional Green's function $G_k(r, r')$ for the Bessel differential operator of order zero; that is, solve

$$\frac{d}{dr} \left(r \frac{dG_k}{dr} \right) + k^2 r G_k = \delta(r - r'), \quad 0 \leq r \leq a,$$

subject to the boundary condition

$$G_k(a, r') = 0.$$

Show that G_k is singular wherever $k = k_n$, where

$$J_0(k_n a) = 0.$$

From the behavior of G_k at this singularity determine the normalization integral

$$\int_0^a r J_0^2(k_n r) dr.$$

[Hint: It is necessary to use both the regular solution to Bessel's equation of order zero, J_0 , and the irregular solution, N_0 . The result of problem 1, as well as the asymptotic behaviors

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right), \quad x \gg 1,$$

$$N_0(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4}\right), \quad x \gg 1,$$

will be helpful.]

4. Find the Green's function for the two-dimensional Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) G(x, y; x', y') = \delta(x - x')\delta(y - y')$$

in the interior of a square of side a , expressed as an eigenfunction expansion. With the origin of coordinates chosen to be one corner of the square, the boundary conditions are as follows (see Fig. 1):

$$\begin{aligned} G(0, y; x', y') &= 0, \\ G(a, y; x', y') &= 0, \\ \frac{\partial}{\partial y} G(x, 0; x', y') &= 0, \\ \frac{\partial}{\partial y} G(x, a; x', y') &= 0. \end{aligned}$$

5. (a) Find the Green's function for Laplace's equation,

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'),$$

in a three-dimensional region lying between the two planes $x = 0$ and $x = a$ as shown in Fig. 2 with the boundary conditions

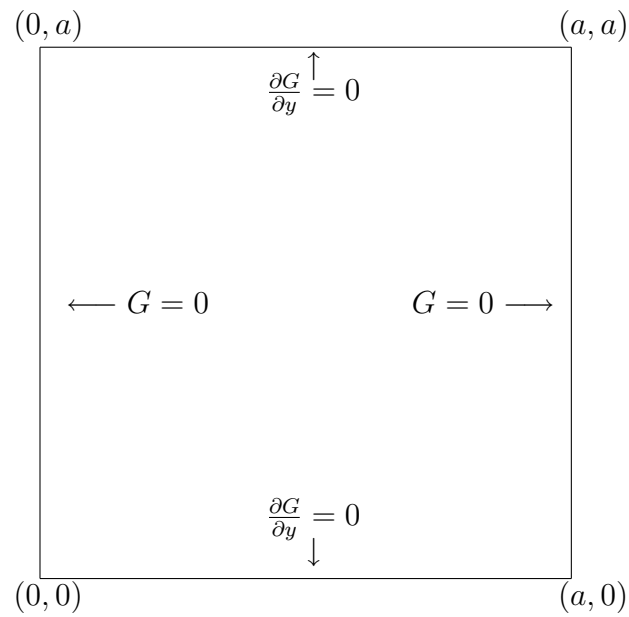


Figure 1: Boundary conditions for the Green's function in Problem 4.

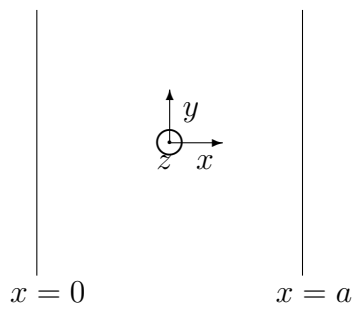


Figure 2: Coordinate system for Problem 5.

$$\begin{aligned} G(0, y, z; x', y', z') &= 0, \\ G(a, y, z; x', y', z') &= 0, \end{aligned}$$

using the following method: Show that G can be written in the form

$$G(\mathbf{r}, \mathbf{r}') = \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z e^{ik_y(y-y')} e^{ik_z(z-z')} g_{k_{\perp}^2}(x, x'),$$

where $k_{\perp}^2 = k_y^2 + k_z^2$, and find $g_{k_{\perp}^2}$ in closed form by using the discontinuity method.

- (b) By examining the singularities of $g_{k_{\perp}^2}(x, x')$ with respect to k_{\perp}^2 find the normalized eigenfunctions and eigenvalues of d^2/dx^2 subject to the homogeneous Dirichlet boundary conditions at $x = 0$ and $x = a$.
- (c) Using the result of (b) and the generating function for the Bessel functions

$$e^{\frac{x}{z}(z-\frac{1}{z})} = \sum_{m=-\infty}^{\infty} z^m J_m(x),$$

prove that

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{2\pi} \int_0^{\infty} dk k J_0(kR) \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x}{a} \sin \frac{n\pi x'}{a}}{k^2 + \left(\frac{n\pi}{z}\right)^2},$$

where $R^2 = (y - y')^2 + (z - z')^2$.